

A complexity dichotomy for poset constraint satisfaction

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Abstract

In this paper we determine the complexity of a broad class of problems that extends the temporal constraint satisfaction problems in [6]. To be more precise we study the problems $\text{Poset-SAT}(\Phi)$, where Φ is a given set of quantifier-free \leq -formulas. An instance of $\text{Poset-SAT}(\Phi)$ consists of finitely many variables x_1, \dots, x_n and formulas $\phi_i(x_{i_1}, \dots, x_{i_k})$ with $\phi_i \in \Phi$; the question is whether this input is satisfied by any partial order on x_1, \dots, x_n or not. We show that every such problem is NP-complete or can be solved in polynomial time, depending on Φ .

All Poset-SAT problems can be formalized as constraint satisfaction problems on reducts of the random partial order. We use model-theoretic concepts and techniques from universal algebra to study these reducts. In the course of this analysis we establish a dichotomy that we believe is of independent interest in universal algebra and model theory.

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1 Introduction

Reasoning about temporal knowledge is a common task in various areas of computer science, for example Artificial Intelligence, Scheduling, Computational Linguistics and Operations Research. In many application temporal constraints are expressed as collections of relations between time points or time intervals. A typical computational problem is then to determine whether such a collection is satisfiable or not.

A lot of research in this area concerns only linear models of time. In particular there exists a complete classification of all satisfiability problems for linear temporal constraints in [6]. However, it has been observed many times that more complex time models are helpful, for instance in the analysis of concurrent and distributed systems or certain planning domains. A possible generalizations is to model time by partial orders (e.g. in [16], [1]).

Some cases of the arising satisfiability problems have already been studied in [13]. We will give a complete classification in this paper. Speaking more formally, let Φ be a set of

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quantifier-free formulas in the language consisting of a binary relation symbol \leq . Then $\text{Poset-SAT}(\Phi)$ asks if constraint expressible in Φ are satisfiable by a partial order or not. We are going to give a full complexity classification of problems of the type $\text{Poset-SAT}(\Phi)$. In particular we are going to show that every such problem is NP-complete or solvable in polynomial time.

The proof of our result is based on a variety of methods and results. A first step is that we give a description of every Poset-SAT problem as *constraint satisfaction problem* over a countably infinite domain, where the constraint relations are first-order definable over the *random partial order*, a well-known structure in model theory.

A helpful result has already been established in the form of the classification of the closed supergroups of the automorphism group of the random partial order in [18]. We extend this analysis to closed transformation monoids. Informally, then our result implies that we can identify three types of Poset-SAT problems: (1) trivial ones (i.e., if there is a solution, there is a constant solution), (2) problems that can be reduced to the problems studied in [6] and (3) CSPs on templates that are model-complete cores.

So we only have to study problems in the third class. The basic method to proceed then is the *universal-algebraic approach* to constraint satisfaction problems. Here, one studies certain sets of operations (known as *polymorphism clones*) instead of analysing the constraints themselves. An important tool to deal with polymorphisms over infinite domains is Ramsey theory. We need a Ramsey result for partially ordered sets from [20] for proving that polymorphisms behave regularly on large parts of their domain. This allows us to perform a more simplified combinatorial analysis.

This paper has the following structure: In Section 2 we introduce some basic notation and show how every Poset-SAT problem is equal to a constraint satisfaction problem on a reduct of the random partial order. In Section 3 we give a brief introduction to the universal-algebraic approach and the methods from Ramsey theory that we need for our classification. Section 4 contains a preclassification, by the analysis of closed transformation monoids containing the automorphism group of the random partial order. This is followed by the actual complexity analysis using the universal algebraic approach. In Section 9 we summarize our results to show the complexity dichotomy for Poset-SAT problems.

We further show in Section 9 that an even stronger dichotomy holds, regarding the question whether certain reducts of the random partial order allow pp-interpretations of all finite structures or not (cf. the discussion in [11] and [22]). In this respect the situation is similar to previous classifications for CSPs where the constraints are first-order definable over the rational order [6], the random graph [10], or the homogeneous binary branching C-relation [17].

2 Preliminaries

In this section we fix some standard terminology and notation. When working with relational structures it is often convenient not to distinguish between a relation and its relational symbol. We will also do so on several occasions, but this should never cause any confusion.

Let \leq always denote a partial order relation, i.e. a binary relation that is reflexive, antisymmetric and transitive. Let $<$ be the corresponding strict order defined by $x < y \wedge x \neq y$. Let $x \perp y$ denote the incomparability relation defined by $\neg(x \leq y) \wedge \neg(y \leq x)$. Sometimes we will write $x < y_1 \cdots y_n$ for the conjunction of the formulas $x < y_i$ for all $1 \leq i \leq n$. Similarly we will use $x \perp y_1 \cdots y_n$ if $x \perp y_i$ holds for all $1 \leq i \leq n$.

2.1 Poset-SAT(Φ)

Let $\phi(x_1, \dots, x_n)$ be a formula in the language that only consists of the binary relation symbol \leq . Then we say that $\phi(x_1, \dots, x_n)$ is *satisfiable* if there exists a partial order $(A; \leq)$ with elements a_1, \dots, a_n such that $\phi(a_1, \dots, a_n)$ holds in $(A; \leq)$. In this case we call $(A; \leq)$ a *solution* to ϕ .

Let $\Phi = \{\phi_1, \phi_2, \dots, \phi_k\}$ be a finite set of quantifier free \leq -formulas. Then the poset satisfiability problem Poset-SAT(Φ) is the following computational problem:

Poset-SAT(Φ):

INSTANCE: A finite set of variables $\{x_1, \dots, x_n\}$ and a finite set of formulas Ψ that is obtained from $\phi \in \Phi$, by substituting the variables of ϕ by variables from $\{x_1, \dots, x_n\}$

QUESTION: Is there a partial order $(A; \leq)$ that is a solution to all formulas in Ψ ?

► **Example 1.** An instance of Poset-SAT($\{<\}$) is given by variables in $\{x_1, \dots, x_n\}$ and formulas in Ψ of the form $x_i < x_j$. The question is, if there is partial order on $\{x_1, \dots, x_n\}$ that satisfies all formulas $x_i < x_j$ in Ψ . It is easy to see that such a partial order always exists if Ψ does not contain formulas $x_{i_1} < x_{i_2}, \dots, x_{i_{n-1}} < x_{i_n}, x_{i_n} < x_{i_1}$. The existence of such “cycles” in Ψ can be verified in polynomial time, thus Poset-SAT($\{<\}$) is tractable.

Every partial order can be extended to a total order. Therefore there is a solution to I if and only if there is a totally ordered solution to I . So Poset-SAT($\{<\}$) is the same problem as the corresponding *temporal constraint satisfaction problem* Temp-SAT($\{<\}$), i.e. the question if there is a total order that is a solution to the input.

► **Example 2.** We define the betweenness relation $\text{Betw}(x, y, z) := (x < y \wedge y < z) \vee (z < y \wedge y < x)$. Again an instance of Poset-SAT($\{\text{Betw}\}$) is accepted if and only if it is accepted by Temp-SAT($\{\text{Betw}\}$). It is known that Temp-SAT($\{\text{Betw}\}$) is NP-complete, so Poset-SAT($\{\text{Betw}\}$) is NP-complete. We remark that if every relation in Φ is positively definable in $<$ that then Poset-SAT(Φ) is the same problem as Temp-SAT(Φ). All temporal constraint satisfaction problems are classified in [6].

► **Example 3.** Let every formula $\phi(x_1, \dots, x_n)$ in Φ be a \leq -Horn formula, that is a formula of the form

$$x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \dots \wedge x_{i_k} \leq x_{j_k} \rightarrow x_{i_{k+1}} \leq x_{j_{k+1}} \text{ or } \\ x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \dots \wedge x_{i_k} \leq x_{j_k} \rightarrow \text{'false'}$$

All problems of this “Horn-type” are tractable. One can see this by giving an algorithm based on the resolution rule. We discuss this class of tractable problems in Section 5.1.

► **Example 4.** We define $\text{Cycl}(x, y, z)$ by

$$\text{Cycl}(x, y, z) := (x < y \wedge y < z) \vee (z < x \wedge x < y) \vee (y < z \wedge z < x) \vee \\ (x < y \wedge x \perp z \wedge y \perp z) \vee (y < z \wedge x \perp y \wedge x \perp z) \vee (z < x \wedge y \perp z \wedge y \perp x).$$

We will see in Section 8 that Poset-SAT($\{\text{Cycl}\}$) is an NP-complete problem.

Every problem Poset-SAT(Φ) is clearly in NP. We can “guess” a partial order on the variables x_1, \dots, x_n and then checks in polynomial time if this partial order is a solution to the input formulas in Ψ or not. The main result of this paper is to give a full classification of the computational complexity, showing the following dichotomy:

► **Theorem 5.** Let Φ be a finite set of quantifier-free \leq -formulas. The problem Poset-SAT(Φ) is in P or NP-complete.

2.2 Poset-SAT(Φ) as CSP

In this section we are going to show that every Poset-SAT problem can be translated into a *constraint satisfaction problem* or *CSP* over an infinite domain. This reformulation will allow us the use of universal-algebraic and Ramsey-theoretical methods.

Let Γ be a relational structure with signature $\tau = \{R_1, R_2, \dots\}$, i.e. $\Gamma = (D; R_1^\Gamma, R_2^\Gamma, \dots)$ where D is the domain of Γ and $R_i^\Gamma \subset D^{k_i}$ is a relation of arity k_i over D . When Δ and Γ are two τ -structures, then a *homomorphism* from Δ to Γ is a mapping h from the domain of Δ to the domain of Γ such that for all $R \in \tau$ and for all $(x_1, \dots, x_j) \in R^\Delta$ we have $h(x_1, \dots, x_j) \in R^\Gamma$. Injective homomorphisms that also preserve the complement of each relation are called *embeddings*. Bijective embeddings from Δ onto itself are called *automorphisms* of Δ .

Suppose that the signature τ of Γ is finite. Then the constraint satisfaction problem $\text{CSP}(\Gamma)$ is the following decision problem:

CSP(Γ):
INSTANCE: A finite τ -structure Δ
QUESTION: Is there a homomorphism from Δ to Γ ?

We say that Γ is the *template* of the constraint satisfaction problem $\text{CSP}(\Gamma)$.

We now need some terminology from model theory. Let Δ be a structure with domain D . We say a relation $R \subset D^n$ is definable in Δ , if there is a first order formula ϕ in the signature of Δ with free variables (x_1, \dots, x_n) such that $(a_1, \dots, a_n) \in R$ if and only if $\phi(a_1, \dots, a_n)$ holds in Δ . A relational structure Γ with the same domain as Δ is called a *reduct* of Δ if all relations of Γ are definable in Δ .

A structure is called *homogeneous* if every isomorphism between finitely generated substructures can be extended to an automorphism of the whole structure.

Let \mathcal{C} be a class of relational structures of the same signature. We say \mathcal{C} has the *amalgamation property* if for every $A, B, C \in \mathcal{C}$ and all embeddings $u : A \rightarrow B$ and $v : A \rightarrow C$ there is a $D \in \mathcal{C}$ with embeddings $u' : B \rightarrow D$, $v' : C \rightarrow D$ such that $u' \circ u = v' \circ v$. A class \mathcal{C} is called an *amalgamation class* if it has the amalgamation property and is closed under isomorphism and taking induced substructures.

► **Theorem 6** (Fraïssé, see Theorem 7.1.2 in [14]). *Let \mathcal{C} be an amalgamation class that has only countably many non-isomorphic members. Then there is a countable homogeneous structure Γ such that \mathcal{C} is the age of Γ , i.e. the class of all structures that embeds into Γ . The structure Γ , which is unique up to isomorphism, is called the Fraïssé limit of Γ . □*

Since the class of all finite partial orders forms an amalgamation class, it has a Fraïssé limit which is called the *random partial order* or *random poset* $\mathbb{P} = (P; \leq)$. The random poset is a well-studied object in model theory that has a lot of helpful properties. As a homogeneous structure in a finite relational language \mathbb{P} has *quantifier-elimination*, i.e. every formula in \mathbb{P} is equivalent to a quantifier-free formula. Also it is ω -categorical, i.e. all countable structures that satisfy the same first order formulas as \mathbb{P} are isomorphic to \mathbb{P} . Reducts of ω -categorical structures are ω -categorical as well. Therefore \mathbb{P} and all of its reducts are ω -categorical. For further model-theoretic background on ω -categorical and homogeneous structures we refer to [14].

Now let $\Phi = \{\phi_1, \dots, \phi_n\}$ be a finite set of quantifier free \leq -formulas. We associate with Φ the τ -structure $\mathbb{P}_\Phi = (P; R_1, \dots, R_n)$ that we obtain by setting $(a_1, \dots, a_k) \in R_i$ if and

only if $\phi_i(a_1, \dots, a_k)$ holds in \mathbb{P} . We claim that $\text{CSP}(\Gamma_\Phi)$ and $\text{Poset-SAT}(\Phi)$ are essentially the same problem. Let Ψ be an instance of $\text{Poset-SAT}(\Phi)$ with the variables x_1, \dots, x_n . Then we define a structure Δ_Ψ with domain $\{x_1, \dots, x_n\}$ as follows: The relation R_i^Δ contains the tuple $(x_{i_1}, \dots, x_{i_{k_i}})$ if and only if the instance Ψ contains the formula $\phi_i(x_{i_1}, \dots, x_{i_{k_i}})$. It is quite straightforward to see that Ψ is accepted by $\text{Poset-SAT}(\Phi)$ if and only if Δ_Ψ homomorphically maps to \mathbb{P}_Φ .

Conversely let $\Gamma = (P; R_1, \dots, R_n)$ be a reduct of \mathbb{P} . Since \mathbb{P} has quantifier-elimination, every relation R_i in Γ has a quantifier-free definition ϕ_i in \mathbb{P} . With that in mind, we study the problem $\text{Poset-SAT}(\{\phi_1, \dots, \phi_n\})$. Let Δ be an instance of $\text{CSP}(\Gamma)$. Then let Ψ_Δ be an instance of $\text{Poset-SAT}(\Phi)$ where the variables are the points in Δ and $\phi_i(x_1, \dots, x_k) \in \Psi$ if and only if $(x_1, \dots, x_k) \in R_i^\Delta$. Then Δ is accepted by $\text{CSP}(\Gamma)$ if and only if Ψ_Δ is accepted by $\text{Poset-SAT}(\Phi)$.

So by the observations in the paragraphs above the following holds:

► **Proposition 7.** *The problems of the form $\text{Poset-SAT}(\Phi)$ correspond precisely to the problems of the form $\text{CSP}(\Gamma)$, where Γ is a reduct of the random partial order \mathbb{P} . \square*

3 The universal-algebraic approach

We apply the so-called universal-algebraic approach and the Ramsey theoretical methods developed by Bodirsky and Pinsker in [10] to obtain our complexity results. Using the language of universal algebra we can elegantly describe the border between tractability and NP-hardness for CSPs on reducts of the random poset. In this section we give a brief introduction to this approach. For a more detailed introduction we refer to [3] as well as the shorter [21].

3.1 Primitive positive definability

A first-order formula $\phi(x_1, \dots, x_n)$ in the language τ is called *primitive positive* if it is of the form $\exists y_1, \dots, y_k (\psi_1 \wedge \dots \wedge \psi_m)$ where ψ_1, \dots, ψ_m are atomic τ -formulas with free variables from the set $\{x_1, \dots, x_n, y_1, \dots, y_k\}$.

Let Γ be a τ -structure. We then say a relation R is *primitively positive definable* or *pp-definable* in Γ if there is a primitive positive formula $\phi(x_1, \dots, x_n)$ such that $(a_1, \dots, a_n) \in R$ if and only if $\phi(a_1, \dots, a_n)$ holds in Γ .

► **Lemma 8** (Jeavons [15]). *Let Γ be a relational structure in finite language, and let Γ' be the structure obtained from Γ by adding a relation R . If R is primitively positive definable in Γ , then $\text{CSP}(\Gamma)$ and $\text{CSP}(\Gamma')$ are polynomial-time equivalent. \square*

By $\langle \Gamma \rangle_{pp}$ we denote the set of all primitively positive definable relations on Γ . So for two structures Γ and Δ the problems $\text{CSP}(\Gamma)$ and $\text{CSP}(\Delta)$ have the same complexity if $\langle \Gamma \rangle_{pp} = \langle \Delta \rangle_{pp}$. This means that in our analysis we only have to study reduct of the random poset up to primitive positive definability.

3.2 Polymorphism clones

Let Γ be a relational structure with domain D . By Γ^n we denote the direct product of n -copies of Γ . This is, we take a structure on D^n with same signature Γ . Then for n -tuples $\bar{x}^{(1)}, \dots, \bar{x}^{(k)}$ we set that $(\bar{x}^{(1)}, \dots, \bar{x}^{(k)}) \in R$ if and only if $(x_i^{(1)}, \dots, x_i^{(k)}) \in R$ holds in Γ for every coordinate $i \in [n]$.

Then an n -ary operation f is called a *polymorphism* of Γ if f is a homomorphism from Γ^n to Γ . Unary polymorphisms are called *endomorphisms*. For every relation R on D we say f *preserves* R if f is a polymorphism of $(D; R)$. Otherwise we say f *violates* R .

For a given structure Γ the set of all polymorphisms $\text{Pol}(\Gamma)$ contains all the projections $\pi_i^n(x_1, \dots, x_n) = x_i$ and is closed under composition. Every set of operation with these properties is called a *clone* or *function clone* (cf. [24]). $\text{Pol}(\Gamma)$ is called the *polymorphism clone* of Γ . We write $\text{Pol}(\Gamma)^{(k)}$ for the set of k -ary functions in $\text{Pol}(\Gamma)$. We write $\text{End}(\Gamma)$ for the monoid consisting of all endomorphisms of Γ .

The clone $\text{Pol}(\Gamma)$ is furthermore *locally closed* in the following sense: Let $k > 1$ be arbitrary and f a k -ary operation on D . If for every finite subset $A \subseteq D^k$ there is a $g \in \text{Pol}(\Gamma)$ with $g \upharpoonright A = f \upharpoonright A$ then also $f \in \text{Pol}(\Gamma)$.

For a set of operation F on D being locally closed is equal to be closed in the topology of pointwise convergence. We write \overline{F} for smallest locally closed set containing F . We say a set of operation F *generates* an operation g if g is in the smallest locally closed clone containing F .

It is of central importance to us that primitive positive definability in ω -categorical (and finite) structures can be characterized by preservation under polymorphisms:

► **Theorem 9** (Bodirsky, Nešetřil [7]). *Let Γ be an ω -categorical structure. Then a relation is pp-definable in Γ , if and only if it is preserved by the polymorphisms of Γ .*

Thus, by Lemma 8 the complexity of $\text{CSP}(\Gamma)$ only depend on the polymorphism clone $\text{Pol}(\Gamma)$ for ω -categorical Γ .

We also need the fact that a relation is not definable in Γ can be described by polymorphisms of bounded arity.

► **Theorem 10** (Bodirsky, Kara [6]). *Let Γ be a relational structure and let R be a k -ary relation that is a union of at most m orbits of $\text{Aut}(\Gamma)$ on D^k . If Γ has a polymorphism f that violates R , then Γ also has an at most m -ary polymorphism that violates R . □*

3.3 Structural Ramsey theory

We apply Ramsey theory to analyse certain regular behaviour of polymorphisms. This approach was invented by Bodirsky and Pinsker and has been proven to be useful in several other classification results (e.g. [6], [10], [17]). We only give a brief introduction how Ramsey theory helps us to study polymorphism clones, a detailed introduction can be found in [8].

Let Γ and Δ be two finite structures of the same signature. Then $(\frac{\Delta}{\Gamma})$ denotes all the substructures of Δ that are isomorphic to Γ . We write $\Theta \rightarrow (\frac{\Delta}{\Gamma})_k^\Gamma$ if for every coloring $\kappa : (\frac{\Theta}{\Gamma}) \rightarrow [k]$ there is an isomorphic copy Δ' of Δ such that κ is monochromatic on $(\frac{\Delta'}{\Gamma})$.

Let \mathcal{C} be a category of structures with the same signature that is closed under taking isomorphic copies and substructures. Then \mathcal{C} is said to be a *Ramsey class* if for every $k \geq 1$ and every $\Gamma, \Delta \in \mathcal{C}$ there is a $\Theta \in \mathcal{C}$ such that $\Theta \rightarrow (\frac{\Delta}{\Gamma})_k^\Gamma$.

A homogeneous structure is said to be a *homogeneous Ramsey structure* if its age is a Ramsey class. A structure is called *ordered*, if it contains a total order relation.

If we look at the class of all structures $(A; \leq, \prec)$, where \prec is a linear order that extends \leq , it has the Ramsey property and the amalgamation property (see [20]). Therefore its Fraïssé limit $(P; \leq, \prec)$ is an ordered homogeneous Ramsey structure.

► **Definition 11.** Let Γ and Δ be two structures and f an n -ary operation from the domain of Γ to the domain of Δ . Let A be a subset of Γ . Then f is called *canonical* on A , if for

every integer $s \geq 1$, all automorphisms $\alpha_1, \dots, \alpha_n \in \text{Aut}(\Gamma)$ and tuples $d_1, \dots, d_n \in A^s$ there is an automorphism $\beta \in \text{Aut}(\Delta)$ such that

$$f(\alpha_1(d_1), \dots, \alpha_n(d_n)) = \beta(f(d_1, \dots, d_n)).$$

If f is canonical on the domain of Γ , we say that f is canonical.

We remark that sometimes canonical functions are also defined as functions that preserve model-theoretic types. However, by the theorem of Ryll-Nardzewski, Engeler and Svenonius (confer [14]), the s -types of a countable ω -categorical structure Γ are exactly the orbits of s -tuples under the action of $\text{Aut}(\Gamma)$. Therefore the model theoretical definition is equivalent to the one we gave above, for ω -categorical Γ and Δ . For this reason we are also going to use (s-)orbit and (s-)type synonymously.

Now Ramsey structures allows us to generate functions that are canonical on arbitrary large finite substructures of Γ in the following sense:

► **Theorem 12** (Lemma 19 in [8]). *Let Γ be an ordered homogeneous Ramsey structure with domain D and let $f : D^l \rightarrow D$. Then for every finite subset A of Γ there are automorphisms $\alpha_1, \dots, \alpha_l \in \text{Aut}(\Gamma)$ such that $f \circ (\alpha_1, \dots, \alpha_l)$ is canonical on A^l .* □

We can refine this statement by additionally fixing constants. Let c_1, \dots, c_n be elements of the domain of Γ . Then $(\Gamma, c_1, \dots, c_n)$ denotes the structure that we obtain by extending Γ by the constants c_1, \dots, c_n .

► **Theorem 13** (Lemma 27 in [8]). *Let Γ be an ordered homogeneous Ramsey structure with domain D . Let $c_1, \dots, c_n \in D$ and $f : D^l \rightarrow D$. Then $\{f\} \cup \text{Aut}(\Gamma, c_1, \dots, c_n)$ generates a function g that is canonical as operation from $(\Gamma, c_1, \dots, c_n)$ to Γ and satisfies $f \upharpoonright \{c_1, \dots, c_n\} = g \upharpoonright \{c_1, \dots, c_n\}$.* □

By the *behaviour* of a canonical function $f : \Delta \rightarrow \Lambda$ we denote the set of all tuples (p, q) where p is an s -type of Δ , q is a s -type of Λ and for every tuple \bar{a} of type p the image $f(\bar{a})$ has type q in Λ . So we can regard the behaviour of a canonical function as a function from the types of Δ to the types of Λ . Let A be a subset of the domain of Δ . We call the behaviour of $f \upharpoonright A$ the *behaviour of f on A* . We say a function $f : \Delta \rightarrow \Lambda$ *behaves like* $g : \Delta \rightarrow \Lambda$ (on A) if their behaviour (on A) is equal.

3.4 Model-complete cores

Let Δ and Γ be to structures with the same signature. We say Δ is *homomorphically equivalent* to Γ if there is a homomorphisms from Δ to Γ and a homomorphism from Γ to Δ . By definition, the constraint satisfaction problems $\text{CSP}(\Delta)$ and $\text{CSP}(\Gamma)$ encode the same computational problem for homomorphically equivalent structures Δ and Γ . By homomorphic equivalence we can find for every CSP a template with some nice model-theoretical properties.

A structure Δ is called a *core* if every endomorphism e of Δ is a self embedding. A structure is called *model-complete* if every formula in its first order theory is equivalent to an existential formula. Note that endomorphisms preserve existential positive formulas, and embeddings preserve existential formulas. In the case of ω -categorical structures also the opposite holds. Therefore we have:

► **Lemma 14** (Lemma 13 in [9]). *A countable ω -categorical structure Γ is a model-complete core if and only if the endomorphism monoid of Γ is generated by $\text{Aut}(\Gamma)$. In this case, every definable relation in Γ is also definable by existential positive formulas.* □

Now every CSP with ω -categorical template can be reformulated as a CSP on a template with model-complete core by the following theorem:

► **Theorem 15** (Theorem 16 from [2]). *Every ω -categorical structure Γ is homomorphically equivalent to a model-complete core which is unique up to isomorphism. This core is ω -categorical or finite.* \square

So an important step in analyzing the complexity of $\text{CSP}(\Gamma)$ is to identify the model-complete core of Γ . By the following theorem of Bodirsky the complexity of the CSP of a core does not increase if we add finitely many constants.

► **Theorem 16** (Theorem 19 from [2]). *Let Γ be a model-complete ω -categorical or finite core, and let c be an element of Γ . Then $\text{CSP}(\Gamma)$ and $\text{CSP}(\Gamma, c)$ have the same complexity, up to polynomial time.* \square

3.5 Primitive positive interpretations

A tool to compare the complexity of CSPs of structures Δ and Γ of possibly different domains and signatures are interpretations. We say Δ is *pp-interpretable* in Γ if there is a $n \geq 1$ and a partial map $I : \Gamma^n \rightarrow \Delta$ such that

- I is surjective,
- the domain of I is pp-definable in Γ ,
- the preimage of the equality relation in Δ is pp-definable in Γ ,
- the preimage of every relation in Δ is pp-definable in Γ .

Then the following result holds:

► **Lemma 17** (Theorem 5.5.6 in [3]). *If Δ is pp-interpretable in Γ then $\text{CSP}(\Delta)$ can be reduced to $\text{CSP}(\Gamma)$ in polynomial time.*

The positive-1-in-3-SAT problem is a well-studied problem in literature that is known to be NP-complete [23]. It can be written as $\text{CSP}(\{0, 1\}; 1\text{IN}3)$ with $1\text{IN}3 := \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. In practice we often show the NP-completeness of $\text{CSP}(\Gamma)$ by finding a pp-interpretation of $(\{0, 1\}; 1\text{IN}3)$ in Γ .

As pp-definability, also pp-interpretation can be translated into the language of clones. A *clone homomorphism* $\xi : \text{Pol}(\Gamma) \rightarrow \text{Pol}(\Delta)$ is a map that preserves arities such that

- $\xi(\pi_i^n) = \pi_i^n$ for all projections,
- $\xi(f \circ (g_1, \dots, g_n)) = \xi(f) \circ (\xi(g_1), \dots, \xi(g_n))$.

For ω -categorical or finite structures Δ and Γ it is known that Δ is pp-interpretable in Γ if and only if there exists a continuous *clone homomorphism* from $\text{Pol}(\Gamma)$ onto $\text{Pol}(\Delta)$.

► **Theorem 18** (Bodirsky, Pinsker [11]). *Let Γ be finite or ω -categorical and Δ be finite. Then Γ has a primitive positive interpretation in Δ if and only if $\text{Pol}(\Gamma)$ has a continuous clone homomorphism to $\text{Pol}(\Delta)$.* \square

The polymorphism clone of $(\{0, 1\}; 1\text{IN}3)$ consist only of the projections π_i^2 on the two element set $\{0, 1\}$. We call it the *projection clone 1*. So $(\{0, 1\}; 1\text{IN}3)$ has a pp-interpretation in the ω -categorical structure Γ if and only if there is a continuous clone homomorphism $\xi : \text{Pol}(\Gamma) \rightarrow \mathbf{1}$. By Theorem 18 then *every* finite structure Δ has a pp-interpretation in Γ , since it is easy to see that there is always a continuous clone homomorphism from $\mathbf{1}$ to $\text{Pol}(\Delta)$.

We summarize that for finite or ω -categorical and finite structures Δ and Γ the complexity of $\text{CSP}(\Delta)$ reduces to $\text{CSP}(\Gamma)$ in the following cases:

1. Δ is pp-interpretable in Γ .
2. Δ is the model-complete core of Γ .
3. Γ is a model-complete core and Δ is obtained by adding finitely many constants to the signature of Γ .

By some recent results of Barto, Pinsker we can also express this fact nicely by polymorphism clones.

► **Theorem 19** (Theorem 1.4 in [22]). *Let Γ be finite or ω -categorical and let Δ be its model-complete core. Then the following are equivalent:*

1. *Every finite structure has a pp-interpretation in some extension of Δ by finitely many constants.*
2. *There is a continuous clone homomorphism $\text{Pol}(\Delta, c_1, \dots, c_n) \rightarrow \mathbf{1}$ for some constants $c_1, \dots, c_n \in \Delta$.*
3. *There is a clone homomorphism $\text{Pol}(\Delta, c_1, \dots, c_n) \rightarrow \mathbf{1}$ for some $c_1, \dots, c_n \in \Delta$.*
4. *$\text{Pol}(\Delta)$ contains no pseudo-Siggers operation, i.e. a 6-ary operation s such that*

$$\alpha s(x, y, x, z, y, z) = \beta s(y, x, z, x, z, y)$$

for some unary $\alpha, \beta \in \text{Pol}(\Delta)$. □

Note that (4) shows that the question whether $(\{0, 1\}; 1\text{IN}3)$ is pp-interpretable in any $(\Delta, c_1, \dots, c_n)$ only depends on algebraic but not the topological properties of $\text{Pol}(\Delta)$.

4 A pre-classification by model-complete cores

In this section we start our analysis of reducts of the random partial order $\mathbb{P} = (P; \leq)$. Our aim is to determine the model-complete core for every reduct Γ of \mathbb{P} . Therefore we are going to study all possible endomorphism monoids $\text{End}(\Gamma) \supseteq \text{Aut}(\mathbb{P})$. Part of the work was already done in [18] where all the automorphism groups $\text{Aut}(\Gamma) \supseteq \text{Aut}(\mathbb{P})$ were determined. Several parts of our proof are very similar to the group case; at that points we are going to directly refer to the corresponding proofs of [18].

If we turn the partial order \mathbb{P} upside-down, then the obtained partial order is again isomorphic to \mathbb{P} . Hence there exists a bijection $\uparrow: P \rightarrow P$ such that for all $x, y \in P$ we have $x < y$ if and only if $\uparrow(y) < \uparrow(x)$. By the homogeneity of \mathbb{P} it is easy to see that the monoid generated by \uparrow and $\text{Aut}(\mathbb{P})$ does not depend on the choice of the function \uparrow .

The class of all finite structures $(A; \leq, F)$, where $(A; \leq)$ is a partial order and F is upwards closed set is an amalgamation class. Its Fraïssé limit is isomorphic to \mathbb{P} with an additional unary relation F . We say F is a *random filter* on \mathbb{P} . Note that F and $I = P \setminus F$ are both isomorphic to the random partial order. Furthermore for every pair $x \in I$ and $y \in F$ either $x < y$ or $x \perp y$ holds.

We define a new order relation $<_F$ on by setting $x <_F y$ if and only if

- $x, y \in F$ and $x < y$ or,
- $x, y \in I$ and $x < y$ or,
- $x \in I, y \in F$ and $x \perp y$.

It is shown in [18] that the resulting structure $(P; <_F)$ is isomorphic to $(P, <)$. We fix a map $\circ_F: P \rightarrow P$ that maps $(P; <)$ isomorphically to $(P, <_F)$. By the homogeneity of \mathbb{P} one can see that the smallest closed monoid generated by \circ and $\text{Aut}(\mathbb{P})$ does not depend on the choice of the random filter F . We fix a random filter F and set $\circ := \circ_F$.

For $B \subseteq \text{Sym}(P)$, let $\langle B \rangle$ denote the smallest closed subgroup of $\text{Sym}(P)$ containing B . For brevity, when it is clear we are discussing supergroups of $\text{Aut}(\mathbb{P})$, we may abuse notation and write $\langle B \rangle$ to mean $\langle B \cup \text{Aut}(\mathbb{P}) \rangle$.

► **Theorem 20** (Theorem 1 from [18]). *Let Γ be a reduct of \mathbb{P} . Then $\text{Aut}(\Gamma) \supseteq \text{Aut}(\mathbb{P})$ is equal to one of the five groups $\text{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circ \rangle$, $\langle \updownarrow, \circ \rangle$ or $\text{Sym}(P)$. \square*

We are going to show the following extension of Theorem 20:

► **Proposition 21.** *Let Γ be a reduct of \mathbb{P} . Then for $\text{End}(\Gamma)$ at least one of the following cases applies:*

1. $\text{End}(\Gamma)$ contains a constant function,
2. $\text{End}(\Gamma)$ contains a function $g_{<}$ that preserves $<$ and maps P onto a chain,
3. $\text{End}(\Gamma)$ contains a function g_{\perp} that preserves \perp and maps P onto an antichain,
4. The automorphism group $\text{Aut}(\Gamma)$ is dense in $\text{End}(\Gamma)$, i.e. Γ is a model-complete core. So by the classification in Theorem 20, $\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circ \rangle$, $\langle \updownarrow, \circ \rangle$ or $\text{Sym}(P)$ in the space of all functions P^P .

Before we start with the proof Proposition 21 we want to point out its relevance for the complexity analysis of the CSPs on reducts of \mathbb{P} .

Constraint satisfaction problems on reducts of $(\mathbb{Q}; <)$ are called *temporal satisfaction problems*. The CSPs on reducts of a countable set with a predicate for equality $(\omega; =)$ are called *equality satisfaction problems*. For both classes a full complexity dichotomy is known, see [6] and [5]. As a corollary of Proposition 21 we get the following pre-classification of CSPs reducing all the cases where Γ is not a model-complete core to temporal or equality satisfaction problems:

► **Corollary 22.** *Let Γ be a reduct of \mathbb{P} . Then one of the following holds*

1. $\text{CSP}(\Gamma)$ is trivial;
2. The model-complete core of Γ is a reduct of $(\omega; =)$,
so $\text{CSP}(\Gamma)$ is equal to an equality satisfaction problem;
3. The model-complete core of Γ is a reduct of $(\mathbb{Q}; <)$,
so $\text{CSP}(\Gamma)$ is equal to a temporal satisfaction problem;
4. $\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circ \rangle$ or $\langle \updownarrow, \circ \rangle$.

Proof. If there is a constant function in $\text{End}(\Gamma)$, then $\text{CSP}(\Gamma)$ accepts every instance, so we are in the first case. So let $\text{End}(\Gamma)$ contain no constants.

Assume that $g_{\perp} \in \text{End}(\Gamma)$. Since g_{\perp} preserves \perp , the image of $(P; \perp)$ under g_{\perp} is isomorphic to a countable antichain, or in other word, a countable set ω with a predicate for inequality $(\omega; \neq)$. Thus, for every reduct of Γ the image $g_{\perp}(\Gamma)$ can be seen as a reduct of $(\omega; \neq)$. Now clearly Γ and $g_{\perp}(\Gamma)$ are homomorphically equivalent. It is shown in [5] that every reduct of $(\omega; \neq)$ without constant endomorphisms is a model-complete core. So we are in the second case.

Now assume that $g_{<} \in \text{End}(\Gamma)$ but $g_{\perp} \notin \text{End}(\Gamma)$. Since $g_{<}$ preserves $<$ and is a chain, the image of $(P; <)$ under $g_{<}$ has to be isomorphic to the rational order $(\mathbb{Q}; <)$. Thus for every reduct of Γ the image $g_{<}(\Gamma)$ can be seen as a reduct of \mathbb{Q} . Now clearly Γ and $g_{<}(\Gamma)$ are homomorphically equivalent. It is shown in [6] that the model-complete core of every reduct of $(\mathbb{Q}, <)$ is either trivial, definable in (ω, \neq) or the reduct itself. So we are in the third case.

Note that also in the case where $\text{End}(\Gamma) = \overline{\text{Sym}(P)}$ we have that $e_{\perp} \in \text{End}(\Gamma)$. So by Proposition 21 we are only left with the cases where $\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circ \rangle$ or $\langle \updownarrow, \circ \rangle$. ◀

Let us define the following relations on P :

$$\text{Betw}(x, y, z) := (x < y \wedge y < z) \vee (z < y \wedge y < x).$$

$$\begin{aligned} \text{Cycl}(x, y, z) := & (x < y \wedge y < z) \vee (y < z \wedge z < x) \vee (z < x \wedge x < y) \vee \\ & (x < y \wedge z \perp xy) \vee (y < z \wedge x \perp yz) \vee (z < x \wedge y \perp zx). \end{aligned}$$

$$\text{Par}(x, y, z) := (x \perp yz \wedge y \perp z) \vee (x < yz \wedge y \perp z) \vee (x > yz \wedge y \perp z)$$

$$\begin{aligned} \text{Sep}(x, y, z, t) := & (\text{Cycl}(x, y, z) \wedge \text{Cycl}(y, z, t) \wedge \text{Cycl}(x, y, t) \wedge \text{Cycl}(x, z, t)) \vee \\ & (\text{Cycl}(z, y, x) \wedge \text{Cycl}(t, z, y) \wedge \text{Cycl}(t, y, x) \wedge \text{Cycl}(t, z, x)). \end{aligned}$$

In Lemma 24 we are going to give a description of the monoids $\overline{\langle \uparrow \rangle}$, $\overline{\langle \circ \rangle}$ and $\overline{\langle \uparrow, \circ \rangle}$ as endomorphism monoids with the help of the above relations. We remark that Cycl and Par describes the orbits triples under $\langle \circ \rangle$ and Sep describes the orbit of a linearly ordered 4-tuple under $\langle \uparrow, \circ \rangle$.

► **Lemma 23.** *The incomparability relation \perp is pp-definable in $(P; <, \text{Cycl})$ and Par is pp-definable in $(P; \text{Cycl})$.*

Proof. To proof the first part of the lemma, let

$$\begin{aligned} \psi(x, y, a, b, c, d) := & x < a < c \wedge x < b < d \wedge y < c \wedge y < d \wedge \text{Cycl}(x, a, y) \wedge \text{Cycl}(x, b, y) \\ & \wedge \text{Cycl}(y, c, b) \wedge \text{Cycl}(y, d, a) \wedge \text{Cycl}(b, d, c) \wedge \text{Cycl}(a, c, d). \end{aligned}$$

We claim that $x \perp y$ is equivalent to $\exists a, b, c, d \psi(x, y, a, b, c, d)$. It is not hard to verify that $x \perp y$ implies $\exists a, b, c, d \psi(x, y, a, b, c, d)$. For the other direction note that $\psi(x, y, a, b, c, d)$ implies that $x \neq y$ because $\text{Cycl}(x, a, y)$ is part of the conjunction ψ .

Let us assume that $x < y$ and $\psi(x, y, a, b, c, d)$ holds for some elements $a, b, c, d \in P$. Then $\text{Cycl}(x, a, y)$ implies that $a < y$, symmetrically we have $b < y$. Since $y < c, d$ we have that $a < d$ and $b < c$. Then $\text{Cycl}(b, d, c)$ implies $d < c$ and $\text{Cycl}(a, c, d)$ implies $c < d$, which is a contradiction.

Now assume that $y < x$ and $\psi(x, y, a, b, c, d)$ holds for some elements $a, b, c, d \in P$. Then we have $y < a, b$ by the transitivity of the order. Then $\text{Cycl}(y, c, b)$ implies $c < b$ and $\text{Cycl}(y, d, a)$ implies $d < a$. But this leads to the contradiction $a < c < b$ and $b < d < a$.

For the second part of the lemma let $s, t \in P$ be two elements with $s < t$. Then the set $X = \{x \in P : s < x < t\}$ is pp-definable in $(P; \text{Cycl}, s, t)$ by the formula $\phi(x) := \text{Cycl}(s, x, t)$. By a back-and-forth argument one can show the two structures $(X; \leq)$ and $(P; \leq)$ are isomorphic. The order relation, restricted to X is also pp-definable in $(P; \text{Cycl}, s, t)$ by the equivalence

$$y <_X z \leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Cycl}(y, z, t).$$

Since \perp is pp-definable in $(P; <, \text{Cycl})$, we have that its restriction to X has a pp-definition in $(P; \text{Cycl}, s, t)$. Therefore also the relation $R = \{(x, y, z) \in X^3 : x \perp y \wedge x \perp z \wedge z \perp y\}$ is pp-definable in $(P; \text{Cycl}, s, t)$. Let $\phi(s, t, u, v, w)$ be a primitive positive formula defining R .

We claim that $\exists x, y \phi(x, y, u, v, w)$ is equivalent to $(u, v, w) \in \text{Par}$. Let $(u, v, w) \in \text{Par}$. The relation Par describes the orbit of a 3-element antichain under the action of $\langle \circ \rangle \subseteq \text{End}(P; \text{Cycl})$. So we can assume that (u, v, w) is a 3-antichain, otherwise we take an image under a suitable function from $\langle \circ \rangle$. Now let us take elements $s < t$ such that $s < uvw$ and $uvw < t$. Then clearly $\psi(s, t, u, v, w)$ has to hold.

Conversely let (s, t, u, v, w) be a tuple such that $\psi(s, t, u, v, w)$ holds. We can assume that $s < t$ (otherwise we take the image of (s, t, u, v, w) under a suitable function in $\langle \circ \rangle$). By what we proved above, (u, v, w) is antichain, hence it satisfies Par . ◀

► **Lemma 24.**

1. $\text{End}(P; <, \perp) = \overline{\text{Aut}(\mathbb{P})}$
2. $\text{End}(P; \text{Betw}, \perp) = \overline{\langle \uparrow \rangle}$
3. $\text{End}(P; \text{Cycl}) = \overline{\langle \circ \rangle}$
4. $\text{End}(P; \text{Sep}) = \overline{\langle \uparrow, \circ \rangle}$

Proof.

1. Clearly $\text{Aut}(\mathbb{P}) \subseteq \text{End}(P; <, \perp)$. For the other inclusion let $f \in \text{End}(P; <, \perp)$. Let $A \subseteq P$ be an arbitrary finite set. The restriction of f to a finite subset $A \subseteq P$ is an isomorphism between posets. By the homogeneity of \mathbb{P} there is an automorphism $\alpha \in \text{Aut}(\mathbb{P})$ such that $f \upharpoonright A = \alpha \upharpoonright A$.
2. Since \uparrow preserves Betw and \perp , we know that $\overline{\langle \uparrow \rangle} \subseteq \text{End}(P; \text{Betw}, \perp)$ holds. For the opposite inclusion let $f \in \text{End}(P; \text{Betw}, \perp)$. If f preserves $<$, then $f \in \text{End}(P; <, \perp)$ and we are done. Otherwise there is a pair of elements $c_1 < c_2$ with $f(c_1) > f(c_2)$. Let $d_1 < d_2$ be an other pair of points in P . Then there are $a_1, a_2 \in P$ such that $c_1 < c_2 < a_1 < a_2$ and $d_1 < d_2 < a_1 < a_2$. Since f preserves Betw, $f(a_1) > f(a_2)$ holds and hence also $f(d_1) > f(d_2)$. So f inverts the order, while preserving \perp . Therefore $\uparrow \circ f \in \text{End}(P; <, \perp)$. We conclude that $f \in \overline{\langle \uparrow \rangle}$.
3. It is easy to see that $\overline{\langle \circ \rangle} \subseteq \text{End}(P; \text{Cycl})$. So let $f \in \text{End}(P; \text{Cycl})$. Clearly f is injective and preserves also the relation $\text{Cycl}'(x, y, z) := \text{Cycl}(y, x, z)$. By Lemma 23, f also preserves the relation Par. Furthermore $\langle \circ \rangle$ is 2-transitive: This can be verified by the fact that for every two elements of P , we can find a $\alpha \in \text{Aut}(\mathbb{P})$ that map one element to the random filter F and the other element to $P \setminus F$. So also $\text{End}(P; \text{Cycl})$ is 2-transitive. It follows that $\text{End}(P; \text{Cycl})$ also preserves the negation of Cycl. In other words, f is a self-embedding of $(P; \text{Cycl})$. So, when restricted to a finite $A \subset P$, f is a partial isomorphism. By the results in [19] we know that $(P; \text{Cycl})$ is a homogeneous structure. Hence for every finite $A \subset P$ we find an automorphism $\alpha \in \text{Aut}(P; \text{Cycl}) = \langle \circ \rangle$ such that $f \upharpoonright A = \alpha \upharpoonright A$.
4. Let $f \in \text{End}(P; \text{Sep})$. We claim that either f or $\uparrow \circ f$ preserves Cycl. If we can prove our claim we are done by (3). First of all note that $\text{Sep}(x, y, z, u)$ implies $\text{Cycl}(x, y, z) \leftrightarrow \text{Cycl}(y, z, u)$.

Without loss of generality let there be a elements $x, y, z \in P$ with $\text{Cycl}(x, y, z)$ and $\text{Cycl}(f(x), f(y), f(z))$, otherwise we look at $\uparrow \circ f$ instead of f . Let (r, s, t) be arbitrary tuple satisfying Cycl.

We can always find elements $a < b < c$ in P that are incomparable with all entries of (x, y, z) and (r, s, t) . Further we can choose elements $u, v \in P$ that are incomparable with (a, b, c) such that $z < u < v$ and $\text{Sep}(x, y, z, u) \wedge \text{Sep}(y, z, u, v)$ holds. This can be done by a case distinction and is left to the reader. By construction we have

$$\text{Sep}(x, y, z, u) \wedge \text{Sep}(y, z, u, v) \wedge \text{Sep}(z, u, v, a) \wedge \text{Sep}(u, v, a, b) \wedge \text{Sep}(v, a, b, c).$$

So we have that $(f(x), f(y), f(z)) \in \text{Cycl}$ if and only if $(f(a), f(b), f(c)) \in \text{Cycl}$. Repeating the same argument for (r, s, t) gives us that $(f(r), f(s), f(t)) \in \text{Cycl}$. So f preserves Cycl. ◀

Recall that we obtain an ordered homogeneous Ramsey structure $(P; \leq, \prec)$ by taking the Fraïssé limit of the class of finite structures $(A; \leq, \prec)$, where $(A; \leq)$ is a partial order on A and \prec an extension of $<$ to a total order. We can regard this structure to be an extension of \mathbb{P} by a total order. By Theorem 13 the following holds:

► **Lemma 25.** *Let $f : P \rightarrow P$ and $c_1, \dots, c_n \in P$ be any points. Then there exists a function $g : P \rightarrow P$ such that*

1. $g \in \overline{\text{Aut}(\mathbb{P}) \cup \{f\}}$.
2. $g(c_i) = f(c_i)$ for $i = 1, \dots, n$.
3. *Regarded as a function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$, g is a canonical function.* \square

Let Γ be a reduct of \mathbb{P} . We are going to study all feasible behaviors of a canonical function $f : (P; \leq, \prec, \bar{c}) \rightarrow (P; \leq)$ when $f \in \text{End}(\Gamma)$. Note that the behaviour of such f only depends on the behaviour on the 2-types because $(P; \leq, \prec, \bar{c})$ is homogeneous and its signature contains at most 2-ary relation symbols. Since there are only finitely many 2-types, the study of all possible behaviors of such canonical functions is a combinatorial problem. We introduce the following notation:

► **Notation 26.** *Let A, B be definable subsets of \mathbb{P} and let $\phi_1(x, y), \dots, \phi_n(x, y)$ be formulas. We let $p_{A, B, \phi_1, \dots, \phi_n}(x, y)$ denote the (partial) type determined by the formula $x \in A \wedge y \in B \wedge \phi_1(x, y) \wedge \dots \wedge \phi_n(x, y)$. Using this notation, we can describe the 2-types of $(P; \leq, \prec, \bar{c})$. They are all of the form $p_{X, Y, \phi, \psi} = \{(a, b) \in P^2 : a \in X, b \in Y, \phi(a, b) \text{ and } \psi(a, b)\}$, where X and Y are 1-types, $\phi \in \{=, <, >, \perp\}$ and $\psi \in \{=, \prec, \succ\}$.*

Let X, Y be two distinct infinite 1-types of $(P; \leq, \prec, \bar{c})$. We write $X \perp_{\prec} Y$ if there are pairs $(x, y), (x', y') \in X \times Y$ with $x < y$ and $x' \perp y'$.

When it is convenient for us we will abuse notation and write \bar{c} to describe the set containing all entries of the tuple \bar{c} .

► **Observation 27.** *The structure $(P; \leq, \prec, \bar{c})$ is a homogeneous structure. If X is an 1-type of $(P; \leq, \prec, \bar{c})$ with infinite elements, then $(X; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$ itself. This can be seen by a back-and-forth argument. Similarly, if X and Y are 1-types of $(P; \leq, \prec, \bar{c})$ with infinite elements such that $X \perp_{\prec} Y$ holds, then $X \cup Y$ is isomorphic to $(P; \leq)$ with X being a random filter. If we define $X \leq Y \leftrightarrow \exists (x, y) \in X \times Y (x \leq y)$ we get a partial order on the 1-types of $(P; \leq, \prec, \bar{c})$ (confer Lemma 18 of [18]). But note that the 1-types of $(P; \leq, \prec, \bar{c})$ are not necessarily linearly ordered by \prec : There can be infinite 1-types X, Y and $(x, y), (x', y') \in X \times Y$ with $x \prec y$, $x \perp y$ and $y' \prec x'$, $x' \perp y'$.*

In the following lemmas let Γ be always be a reduct of \mathbb{P} and let $f \in \text{End}(\Gamma)$ be a canonical function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$.

► **Lemma 28.** *Let X be a 1-type of $(P; \leq, \prec, \bar{c})$ with infinite elements. Then f behaves like id or \uparrow on X , otherwise $\text{End}(\Gamma)$ contains a constant function, $g_{<}$ or g_{\perp} .*

Proof. Note that $(X; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$. Then we can prove the statement with the same arguments as in Lemma 8 of [18]. \blacktriangleleft

► **Lemma 29.** *Let X, Y two infinite 1-types of $(P; \leq, \prec, \bar{c})$ with $X \perp_{\prec} Y$. Assume f behaves like id on X . Then f behaves like id or \circ_X on $X \cup Y$, otherwise $\text{End}(\Gamma)$ contains a constant function, $g_{<}$ or g_{\perp} .*

Proof. Assume that f does not contains a constant function, $g_{<}$ or g_{\perp} . Note that the union of X and Y is isomorphic to \mathbb{P} and X is a random filter of $X \cup Y$. By following the arguments of Lemma 22 in [18] one can show that we only have the two possibilities that

1. $f(p_{X, Y, <}) = p_{<}$ and $f(p_{X, Y, \perp, \prec}) = p_{\perp}$ or
2. $f(p_{X, Y, <}) = p_{\perp}$ and $f(p_{X, Y, \perp, \prec}) = p_{>}$.

By Lemma 28 we may assume that f behaves like id or \uparrow on Y . But if f behaves like \uparrow on Y , the image of $y_1, y_2 \in Y$ and $x \in X$ with $x \prec y_1 < y_2$, $x \perp y_1$ and $x < y_2$ would be a non partially ordered set. So if the type $p_{X,Y,\perp,\succ}$ is empty, f behaves like id or \circlearrowleft_X on $X \cup Y$ and we are done.

If $p_{X,Y,\perp,\succ}$ is not empty, there are $x \in X$ and $y \in Y$ with $x \succ y$ and $x \perp y$. We claim that in this case $f(p_{X,Y,\perp,\succ}) = f(p_{X,Y,\perp,\prec})$. We only prove this claim for (1), the proof for (2) is the same.

Assume that $f(p_{X,Y,\perp,\succ}) = p_{<}$. Then let $x' \in X$ be an element such that $y \prec x'$ and $x < x'$ and $y \perp x'$. The fact that such an element exists can be verified by checking that the extension of $\{x, y\} \cup \bar{c}$ by such an element x' still lies in the age of $(P; \leq, \prec, \bar{c})$. By our assumption we then have $f(x) < f(x') < f(y)$, which contradicts to $f(x) \perp f(y)$.

Now assume that $f(p_{X,Y,\perp,\succ}) = p_{>}$. Then let $x' \in X$ be such that $x \prec y \prec x'$ and $x < y$ and $x' \perp xy$. Again the fact that x' exists can be verified by the homogeneity of $(P; \leq, \prec, \bar{c})$. Then $f(x) < f(y) < f(x')$, which contradicts to $f(x') \perp f(x')$. \blacktriangleleft

► **Lemma 30.** *Either f behaves like id or \uparrow on every single 1-type or $\text{End}(\Gamma)$ contains a constant function, $g_{<}$ or g_{\perp} .*

Proof. For every two infinite orbits $X < Y$ there is a infinite orbit Z with $X \perp_{<} Z$ and $Z \perp_{<} Y$. For every two infinite orbits $X \perp Y$ there is an infinite orbit Z with $X < Z$ and $Y < Z$. So this statement holds by Lemma 29. (cf Lemma 23 of [18]) \blacktriangleleft

► **Lemma 31.** *Assume $\text{End}(\Gamma)$ does not contains constant functions, $g_{<}$ or g_{\perp} . Then there is a $g \in \overline{\langle \circlearrowleft, \uparrow \rangle} \cap \text{End}(\Gamma)$ such that $g \circ f$ is canonical from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$ and behaves like id on every set $(P \setminus \bar{c}) \cup \{c\}$, with $c \in \bar{c}$.*

Proof. By Lemma 30, f behaves like id or \uparrow on every infinite orbit. Without loss of generality we can assume that the first case holds, otherwise consider $\uparrow \circ f$.

Let $X \perp_{<} Y$, $Y \perp_{<} Z$ and $X \perp_{<} Z$ or $X < Z$. If f behaves like id on $X \cup Y$ and $Y \cup Z$ it also has to behave like id on $X \cup Z$; otherwise the image of a triple $(x, y, z) \in X \times Y \times Z$ with $x < y < z$ would not be partially ordered. Let $X < Z$, $Y < Z$ and $X \perp Z$. Again, if f behaves like id on $X \cup Y$ and $Y \cup Z$ it also has to behave like id on $X \cup Z$, otherwise we get a contradiction.

By Lemma 29 f either behaves like id or like \circlearrowleft_X on the union two orbits $X \perp_{<} Y$. In the second case $\circlearrowleft \in \text{End}(\Gamma)$. The set $A = \{x \in P : y < x \vee y \perp x \text{ for all } y \in f(Y)\}$ is a union of orbits of $\text{Aut}(P; \leq, \prec, \bar{c})$ and a random filter of P . So $\circlearrowleft_A \circ f$ is canonical and behaves like id on $X \cup Y$. Repeating this step finitely many times gives us a function $g \in \langle \circlearrowleft \rangle$ such that $g \circ f$ behaves like id on the union of infinite orbits, by the observations in the paragraph above.

It is only left to show that $g \circ f$ behaves like id between a given constant c in \bar{c} and an infinite orbit X . Assume for example that $c < X$ and $g \circ f(p_{c,X,<}) = p_{\perp}$. Let $A \subseteq P$ with $a \in A$. By homogeneity of \mathbb{P} we find an automorphism of \mathbb{P} that maps a to c and all points that are greater than a to X . If we then apply $g \circ f$ and repeat this process at most $|A|$ -times we can map A to an antichain. Thus $g_{\perp} \in \text{End}(\Gamma)$ which contradicts to our assumption.

Similarly all other cases where $g \circ f$ does not behave like id between c and X contradict our assumptions. We leave the proof to the reader. Hence $g \circ f$ behaves like id everywhere except on \bar{c} . \blacktriangleleft

Now we are ready to proof the main result of the section.

Proof of Proposition 21. Let Γ be a reduct of \mathbb{P} such that $\text{End}(\Gamma)$ does not contains constant functions, $g_{<}$ or g_{\perp} . We show that then $\text{End}(\Gamma)$ is equal to $\text{Aut}(\mathbb{P})$, $\langle \uparrow \rangle$, $\langle \circlearrowleft \rangle$ or $\langle \uparrow, \circlearrowleft \rangle$.

First assume that $\text{End}(\Gamma)$ contains a non injective function. This can be witnessed by constants $c_1 \neq c_2$ and a function $f \in \text{End}(\Gamma)$ with $f(c_1) = f(c_2)$ that is canonical as function $f : (P; \leq, \prec, c_1, c_2) \rightarrow (P; \leq)$. By Lemma 31 we can assume that f behaves like id everywhere except from c_1, c_2 . But this is not possible, since there is a point in $a \in P$ with $a \perp c_1$ but $\neg(a \perp c_2)$. Since $f(c_1) = f(c_2)$ either $<$ or \perp is violated, which contradicts to f behaving like id everywhere except on $\{c_1, c_2\}$. So from now on let $\text{End}(\Gamma)$ only contain injective functions.

Assume $\text{End}(\Gamma)$ violates Sep. This can also be witnessed by a canonical function $f : (P; \leq, \prec, \bar{c}) \rightarrow (P; \leq)$ such that $\bar{c} \in \text{Sep}$ but $f(\bar{c}) \notin \text{Sep}$. By Lemma 31 we can assume that f behaves like id on every set $(P \setminus \bar{c}) \cup \{c\}$, with $c \in \bar{c}$. If there are $c_i < c_j$ with $f(c_i) \perp f(c_j)$ it is easy to see that $\text{End}(\Gamma)$ generates g_\perp which contradicts to our assumptions. If there are $c_i < c_j$ or $c_i \perp c_j$ with $f(c_i) > f(c_j)$ let a be an element of $(P \setminus \bar{c})$ with $a < c_j$ and $a \perp c_i$. Then the image of a, c_i, c_j under f induces a non partially ordered structure - contradiction.

So $\text{End}(\Gamma)$ preserves Sep. By Lemma 24 we know that $\text{End}(\Gamma) \subseteq \overline{\langle \uparrow, \circ \rangle}$. If $\text{End}(\Gamma)$ violates Cycl and Betw or Cycl and \perp we can proof as in the paragraph above that $\text{End}(\Gamma) = \overline{\langle \uparrow, \circ \rangle}$.

Similarly, if $\text{End}(\Gamma)$ preserves Cycl but violates Betw or \perp then $\text{End}(\Gamma) = \overline{\langle \circ \rangle}$.

If $\text{End}(\Gamma)$ preserves Betw and \perp but violates Cycl. Then $\text{End}(\Gamma) = \overline{\langle \uparrow \rangle}$.

Finally, if $\text{End}(\Gamma)$ preserves Betw, \perp and Cycl we have $\text{End}(\Gamma) = \overline{\text{Aut}(\mathbb{P})}$. \blacktriangleleft

5 The case where $<$ and \perp are pp-definable

Throughout the remaining parts of this paper we are going to study the complexity of $\text{CSP}(\Gamma)$ for model-complete reducts Γ of \mathbb{P} . We start with the case where $\text{End}(\Gamma)$ is the topological closure of the automorphism group of \mathbb{P} . In this case the two relations $<$ and \perp are pp-definable by Theorem 10. So throughout this section let Γ be a reduct of \mathbb{P} in which $<$ and \perp are pp-definable. We are first going to discuss the binary part of the $\text{Pol}(\Gamma)$. This will be essential for proving the dichotomy in this case.

► **Observation 32.** *The binary relation $x \stackrel{\perp}{<} y$ defined by $x < y \vee x \perp y$ is equivalent to the primitive positive formula $\exists z (z < y) \wedge z \perp x$. Therefore $x \stackrel{\perp}{<} y$ is pp definable in Γ .*

By $e_<$ we denote an embedding of the structure $(P; <)^2$ into $(P; <)$. Clearly $e_<$ is canonical when regarded as map $e_< : (P; \leq, \prec)^2 \rightarrow (P; \leq)$. It has the following behaviour:

$e_<$	$=$	$<$	$>$	\perp
$=$	$=$	\perp	\perp	\perp
$<$	\perp	$<$	\perp	\perp
$>$	\perp	\perp	$>$	\perp
\perp	\perp	\perp	\perp	\perp

By e_\leq we denote an embedding of $(P; \leq)^2$ into $(P; \leq)$ that is canonical function when regarded as map $e_\leq : (P; \leq, \prec)^2 \rightarrow (P; \leq)$. It has the following behaviour:

e_\leq	$=$	$<$	$>$	\perp
$=$	$=$	$<$	$>$	\perp
$<$	$<$	$<$	\perp	\perp
$>$	$>$	\perp	$>$	\perp
\perp	\perp	\perp	\perp	\perp

5.1 Horn tractable CSPs given by $e_<$ and e_\leq

The two functions $e_<$ or e_\leq are of central interest to us. We will show in this section that if one of them is a polymorphism of Γ , then the problem $\text{CSP}(\Gamma)$ is tractable.

Let Δ and Λ be relational structures of the same signature. We say a map $h : \Delta \rightarrow \Lambda$ is a *strong homomorphism* if $\bar{x} \in R \leftrightarrow h(\bar{x}) \in R$. By $\hat{\Delta}$ we denote the extension of Δ that contains the negation $\neg R$ for every R is in Δ .

► **Theorem 33** (Proposition 14 from [4]). *Let Δ be an ω -categorical structure and let Γ be a reduct of Δ . Suppose $\text{CSP}(\hat{\Delta})$ is tractable. If Γ has a polymorphism that is a strong homomorphism from Δ^2 to Δ , then also Γ is tractable.* \square

By definition $e_<$ is a strong homomorphism from $(P; <)^2 \rightarrow (P; <)$ and e_\leq is a strong homomorphism from $(P; \leq)^2 \rightarrow (P; \leq)$. Let $\not<$ respectively $\not\leq$ denote the negation of the order relation $<$ respectively \leq . One can see that every input to $\text{CSP}(P; <, \not<)$ and $\text{CSP}(P; \leq, \not\leq)$ is accepted as long as it does not contradict to the transitivity of $<$ respectively \leq . But this can be checked in polynomial time, thus the two problems are tractable. So by Theorem 33 every template Γ with polymorphism $e_<$ or e_\leq gives us a tractable problem.

In the following theorem we additionally give a semantic characterization of these tractable problems via Horn formulas. This characterisation works also in the general setting, we refer to [4] for the proof.

► **Theorem 34.** *Let Γ be a reduct of \mathbb{P} . Suppose that $e_\leq \in \text{Pol}(\Gamma)$. Then $\text{CSP}(\Gamma)$ is tractable and every relation in Γ is equivalent to Horn formula in $(P; \leq)$:*

$$\begin{aligned} x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \cdots \wedge x_{i_k} \leq x_{j_k} &\rightarrow x_{i_{k+1}} \leq x_{j_{k+1}} \quad \text{or} \\ x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \cdots \wedge x_{i_k} \leq x_{j_k} &\rightarrow \text{'false'} \end{aligned}$$

Suppose that $e_< \in \text{Pol}(\Gamma)$. Then $\text{CSP}(\Gamma)$ is tractable and every relation in Γ is equivalent to a Horn formula in $(P; <)$, i.e. a formula of the form:

$$\begin{aligned} x_{i_1} \triangleleft_1 x_{j_1} \wedge x_{i_2} \triangleleft_2 x_{j_2} \wedge \cdots \wedge x_{i_k} \triangleleft_k x_{j_k} &\rightarrow x_{i_{k+1}} \triangleleft_{k+1} x_{j_{k+1}} \quad \text{or} \\ x_{i_1} \triangleleft_1 x_{j_1} \wedge x_{i_2} \triangleleft_2 x_{j_2} \wedge \cdots \wedge x_{i_k} \triangleleft_k x_{j_k} &\rightarrow \text{'false'}, \end{aligned}$$

where $\triangleleft_i \in \{<, =\}$ for all $i = 1, \dots, k+1$. \square

5.2 Canonical binary functions on $(P; \leq, \prec)$

A first step in analysing the binary part of $\text{Pol}(\Gamma)$ is to look at the special case of canonical functions. So in the following text we are going to study the behaviour of binary functions $f \in \text{Pol}(\Gamma)$ that are canonical seen as functions from $(P; \leq, \prec)^2$ to $(P; \leq)$. We are going to specify conditions for which $\text{Pol}(\Gamma)$ contains $e_<$ or e_\leq .

► **Definition 35.** Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}$ be a function. Then f is called *dominated on the first argument* if

- $f(x, y) < f(x', y')$ for all $x < x'$ and
- $f(x, y) \perp f(x', y')$ for all $x \perp x'$.

We say f is *dominated* if f or $(x, y) \mapsto f(y, x)$ is dominated on the first argument.

We are going to prove the following theorem:

► **Theorem 36.** *Let Γ be a reduct of \mathbb{P} in which $<$ and \perp are pp-definable. Let $f(x, y) \in \text{Pol}(\Gamma)$ be canonical when seen as a function from $(P; \leq, \prec)^2$ to $(P; \leq)$. Then at least one of the following cases holds:*

- f is dominated
- $\text{Pol}(\Gamma)$ contains $e_{<}$
- $\text{Pol}(\Gamma)$ contains e_{\leq}

First of all we make some general observations for binary canonical functions preserving $<$ and \perp . We are again going to use the notation introduced in Notation 26. Let us fix a function $- : (P; \leq, <) \rightarrow (P; \leq, <)$ such that $x < y \leftrightarrow -y < -x$ holds. It is easy to see that such a function exists.

► **Lemma 37.** *Let $f : (P; \leq, <)^2 \rightarrow (P; \leq)$ be canonical and $f \in \text{Pol}(\Gamma)$. Then the following statements are true:*

1. $f(p_{<}, p_{<}) = p_{<}$, $f(p_{\perp}, p_{\perp}) = p_{\perp}$
2. $f(p, q) = -f(-p, -q)$, for all types p, q .
3. $f(p_{<}, p_{\perp, <})$, $f(p_{<}, p_{\perp, >})$, $f(p_{\perp, >}, p_{<})$ and $f(p_{\perp, <}, p_{<})$ can only be equal to $p_{<}$ or p_{\perp} .
4. At least one of $f(p_{<}, p_{\perp, <})$ and $f(p_{\perp, <}, p_{<})$ is equal to p_{\perp} .
5. At least one of $f(p_{<}, p_{\perp, >})$ and $f(p_{\perp, >}, p_{<})$ is equal to p_{\perp} .
6. It is not possible that $f(p_{<}, p_{>}) = p_{=}$ holds.
7. $f(p_{\perp, <}, p_{<}) = p_{\perp} \rightarrow f(p_{\perp}, p_{=}) = p_{\perp}$

Proof.

1. This is clear, since f is a polymorphism of Γ and hence preserves $<$ and \perp .
2. This is true by definition of $-$.
3. This is true since f preserves the relation $\frac{\perp}{<}$, see Observation 32.
4. Assume $f(p_{<}, p_{\perp, <}) = f(p_{\perp, <}, p_{<}) = p_{<}$. Let $a_1 < a_2 < a_3$ with $a_1 < a_2$, $a_3 \perp a_1 a_2$ and $b_1 < b_2 < b_3$ with $b_2 < b_3$, $b_1 \perp b_2 b_3$. By our assumption $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ holds, which is a contradiction to f preserving \perp .
5. This can be proven similarly to (4).
6. Assume that $f(p_{<}, p_{>}) = p_{=}$ holds. Let $a_1 < a_2 < a_3$ with $a_1 < a_3$, $a_2 < a_3$, $a_1 \perp a_2$ and $b_1 > b_2 > b_3$ with $b_1 > b_3$, $b_2 > b_3$, $b_1 \perp b_2$. Then $f(a_1, b_1) \perp f(a_2, b_2)$ but also $f(a_1, b_1) = f(a_3, b_3) = f(a_2, a_2)$ have to hold, which is a contradiction.
7. Assume that there are $a_1 \perp a_2$ and b such that $f(a_1, b) \leq f(a_2, b)$ holds. Then we take elements a_3 and b' with $a_2 < a_3$, $a_1 \perp a_3$, $a_1 < a_3$ and $b' > b$. Then $f(a_1, b) \leq f(a_2, b) < f(a_3, b')$ holds, which is a contradiction to $f(a_1, b) \perp f(a_3, b')$. ◀

By Lemma 37 (2) we only have to consider pairs of types where the first entry is $p_{=}$, $p_{<}$ or $p_{\perp, <}$ when studying the behaviour of f . Further Lemma 37 implies that $f(x, y) \neq f(x', y')$ always holds for $x \neq x'$ and $y \neq y'$.

► **Lemma 38.** *Let $f \in \text{Pol}(\Gamma)$. Then the following are equivalent:*

1. $f(p_{<}, p_{>}) = p_{<}$
2. $f(p_{<}, p_{\perp, >}) = p_{<}$
3. $f(p_{<}, q) = p_{<}$ for all 2-types q
4. f is dominated in the first argument

Proof. It is clear that the implications (4) \rightarrow (3) \rightarrow (2) and (3) \rightarrow (1) are true.

(1) \rightarrow (3): Let $a_1 < a_2 < a_3$ and $b_1 b_3 < b_2$. Then $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ has to hold regardless if the type of (b_1, b_3) is $p_{\perp, <}$, $p_{\perp, >}$ or $p_{=}$. So $f(p_{<}, q) = p_{<}$ for all 2-types q .

(2) \rightarrow (1): Let $a_1 < a_2 < a_3$ and $b_1 > b_2 > b_3$ with $b_1 > b_3$, $b_2 \perp b_1 b_3$. Then $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ implies $f(a_1, b_1) < f(a_3, b_3)$ and so $f(p_{<}, p_{>}) = p_{<}$.

(3) \rightarrow (4): We have to consider all the pairs of 2-types where the first entry is $p_{\perp, \prec}$. By Lemma 37 (4) and (5) we know that $f(p_{\perp, \prec}, p_{\prec}) = f(p_{\perp, \prec}, p_{\succ}) = p_{\perp}$. From Lemma 37(7) follows that $f(p_{\perp}, p_{=}) = p_{\perp}$.

We want to point out that we did not require f to be canonical; it can be easily verified that all proof steps also work for general binary functions. \blacktriangleleft

► **Lemma 39.** *Let $f : (P; \leq, \prec)^2 \rightarrow (P; \leq)$ be canonical and $f \in \text{Pol}(\Gamma)$. If f is not dominated the following statements are true:*

1. $f(p_{\prec}, p_{\succ}) = f(p_{\prec}, p_{\perp, \succ}) = f(p_{\perp, \prec}, p_{\succ}) = p_{\perp}$.
2. $f(p_{\prec}, p_{=}) = p_{\prec}$ or $f(p_{\prec}, p_{=}) = p_{\perp}$.
3. $f(p_{\perp, \prec}, p_{=}) = p_{\perp}$ or $f(p_{\perp, \prec}, p_{=}) = p_{\prec}$.

Proof.

1. is a direct consequence of Lemma 38.
2. Suppose there are $a_1 < a_2$ and b such that $f(a_1, b) \geq f(a_2, b)$. Then we take elements $a_3, b' \in P$ with $a_2 \perp a_3$, $a_2 \succ a_3$, $a_1 < a_3$ and a $b' > b$. Then $f(a_2, b) \leq f(a_1, b) < f(a_3, b')$ holds, which is a contradiction to $f(a_2, b) \perp f(a_3, b')$.
3. Assume that there are $a_1 \perp a_2$, $a_1 \prec a_2$ and b such that $f(a_1, b) \geq f(a_2, b)$ holds. There are elements a_3 and b' with $a_2 > a_3$, $a_1 \perp a_3$, $a_1 \prec a_3$ and $b' < b$. Then $f(a_2, b) > f(a_3, b')$ and $f(a_1, b) \perp f(a_3, b')$. But this contradicts to our assumption. \blacktriangleleft

► **Definition 40.** Let us say a binary function is \perp -falling, if it has the same behaviour as e_{\prec} respectively e_{\leq} on pairs of partial type (p_{\neq}, p_{\neq}) .

► **Lemma 41.** *Let $f \in \text{Pol}(\Gamma)$ be a canonical function $f : (P; \leq, \prec)^2 \rightarrow (P; \leq)$ of \perp -falling behaviour. Then $\text{Pol}(\Gamma)$ contains e_{\prec} or e_{\leq} .*

Proof. From Lemma 37 (7) follows that $f(p_{\perp}, p_{=}) = p_{\perp}$ and $f(p_{=}, p_{\perp}) = p_{\perp}$. By Lemma 39 we further know that $f(p_{\prec}, p_{=}), f(p_{=}, p_{\prec}) \in \{p_{\perp}, p_{\prec}\}$. So we have to do a simple case distinction:

- If $f(p_{=}, p_{\prec}) = f(p_{\prec}, p_{=}) = p_{\perp}$, then f behaves like e_{\prec} , hence $e_{\prec} \in \text{Pol}(\Gamma)$.
- If $f(p_{=}, p_{\prec}) = p_{\prec}$ and $f(p_{\prec}, p_{=}) = p_{\perp}$, the function $(x, y) \rightarrow f(f(x, y), x)$ has the same behaviour as e_{\prec} , thus $e_{\prec} \in \text{Pol}(\Gamma)$.
- Symmetrically if $f(p_{=}, p_{\prec}) = p_{\perp}$ and $f(p_{\prec}, p_{=}) = p_{\prec}$, the function $(x, y) \rightarrow f(f(y, x), y)$ has the same behaviour as e_{\prec} , thus $e_{\prec} \in \text{Pol}(\Gamma)$.
- If $f(p_{=}, p_{\prec}) = p_{\prec} = f(p_{\prec}, p_{=}) = p_{\prec}$, then f has the same behaviour as e_{\leq} , thus $e_{\leq} \in \text{Pol}(\Gamma)$. \blacktriangleleft

We now give a simple criterium for the existence of a canonical \perp -falling function in $\text{Pol}(\Gamma)$. This criterium will allow us to finish the proof of Theorem 36.

► **Lemma 42.** *Assume that for every $k > 1$, every pair of tuples $\bar{a}, \bar{b} \in P^k$ and every indices $p, q \in [k]$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$ there exists a binary function $g \in \text{Pol}(\Gamma)$ such that $g(a_p, b_p) \perp g(a_q, b_q)$ and for all $i, j \in [k]$:*

1. $a_i < a_j$ implies $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$,
2. $a_i \perp a_j$ implies $g(a_i, b_i) \perp g(a_j, b_j)$.

Then $\text{Pol}(\Gamma)$ contains e_{\prec} and e_{\leq} .

Proof. First we are going to show that for all $\bar{a}, \bar{b} \in P^k$ there is a binary function $f \in \text{Pol}(\Gamma)$ that has \perp -falling on (\bar{a}, \bar{b}) . To be more precise we want to construct an $f \in \text{Pol}(\Gamma)$ such that:

- $f(a_i, b_i) < f(a_j, b_j)$ if $a_i < a_j$ and $b_i < b_j$,
- $f(a_i, b_i) \perp f(a_j, b_j)$ if $a_i < a_j$ and $\neg(b_p \leq b_q)$.
- $f(a_i, b_i) \perp f(a_j, b_j)$ if $a_i \perp a_j$ and $b_i \neq b_j$

We are going to construct f by a recursive argument.

Let $f^{(0)}(x, y) = g^{(0)}(x, y) = x$ and $\bar{a}^{(0)} = f^{(0)}(\bar{a}, \bar{b})$. If already $f^{(0)}$ has the desired properties we set $f(x, y) = f^{(0)}(x, y)$ and are done.

Otherwise, in the $(k+1)$ -th recursion step, we are given a function $f^{(k)}(x, y)$ and a tuple $\bar{a}^{(k)} = f^{(k)}(\bar{a}, \bar{b})$. Let us assume that there are indices p, q with $a_p < a_q$, $\neg(b_p \leq b_q)$ and $a_p^{(k)} < a_q^{(k)}$. Then by our assumption there is a function $g^{(k+1)}(x, y) \in \text{Pol}(\Gamma)$ such that $g^{(k+1)}(a_p^{(k)}, b_p) \perp g^{(k+1)}(a_q^{(k)}, b_p)$. We set $f^{(k+1)}(x, y) = g^{(k+1)}(f^{(k)}(x, y), y)$ and $\bar{a}^{(k+1)} = f^{(k+1)}(\bar{a}, \bar{b})$.

Note that by the properties (1) and (2) of the function g^k the only possible cases for f^k being not \perp -falling is the case above. It is clear that the recursion ends after finitely many steps.

So on every finite subset $X \times Y$ of P^2 we find a \perp -falling function. By a compactness argument there exists a $h \in \text{Pol}(\Gamma)$ that is \perp -falling on P^2 . It remains to show that there is also a canonical \perp -falling function in $\text{Pol}(\Gamma)$.

By Theorem 12 we have that h is canonical on arbitrarily large substructures of P^2 . Let $(F_n)_{n \in \omega}$ be an increasing sequence of finite substructures such that its union is equal to P . Then for every $n \in \omega$ there are $\alpha_1^{(n)}, \alpha_2^{(n)} \in \text{Aut}(\Gamma)$ such that $f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ is canonical on F_n . By thinning out the sequence we can assume that $f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ has the same behaviour for every $n \in \omega$.

Since the behaviour $f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ on all F_n is the same, we can inductively pick automorphisms $\beta_n \in \text{Aut}(\mathbb{P})$ such that $\beta_n \circ f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ agrees with $\beta_{n+1} \circ f \circ (\alpha_1^{(n+1)}, \alpha_2^{(n+1)})$ on F_n . The limit of this sequence is a canonical function in $\text{Pol}(\Gamma)$ with \perp -falling behaviour.

By Lemma 41 we have that $e_{<}$ or e_{\leq} is an element of $\text{Pol}(\Gamma)$. This concludes the proof. ◀

Proof of Theorem 36. Let $f : (P; \leq, \prec)^2 \rightarrow (P; \leq)$ be canonical and $f \in \text{Pol}(\Gamma)$. Let us assume that f is not dominated. By Lemma 39 we know $f(p_{<}, p_{>}) = f(p_{<}, p_{\perp, >}) = f(p_{\perp, <}, p_{>}) = p_{\perp}$.

By Lemma 37 (3) and (4) we have to look at the following cases:

1. $f(p_{<}, p_{\perp, <}) = f(p_{\perp, <}, p_{<}) = p_{\perp}$.
2. $f(p_{<}, p_{\perp, <}) = p_{<}$ and $f(p_{\perp, <}, p_{<}) = p_{\perp}$.
3. $f(p_{<}, p_{\perp, <}) = p_{\perp}$ and $f(p_{\perp, <}, p_{<}) = p_{<}$.

In the first case f has \perp -falling behaviour therefore we are done by Lemma 41.

For the remaining cases we can restrict ourselves to (2), otherwise we take $(x, y) \rightarrow f(y, x)$. From Lemma 37 (7) follows that $f(p_{\perp}, p_{=}) = p_{\perp}$. Thus $f(p_{\perp}, q) = p_{\perp}$ holds for every 2-type q .

We are going to show that then the conditions in Lemma 42 are satisfied. Let $\bar{a}, \bar{b} \in P^k$ be two tuples of arbitrary length k and let $p, q \in [k]$ such that $a_p < a_q$, $b_p \prec b_q$ and $b_p \perp b_q$ hold. Then let $\alpha \in \text{Aut}(\mathbb{P})$ with $\alpha(b_p) \succ \alpha(b_q)$. Such an automorphism exists by the homogeneity of \mathbb{P} . Then we set $g(x, y) = f(x, \alpha(y))$.

Clearly $g(a_p, a_q) \perp g(b_p, b_q)$, since $\alpha(b_p) \succ \alpha(b_q)$. Also the other conditions in Lemma 42 are satisfied, by the properties of f . Therefore $\text{Pol}(\Gamma)$ contains $e_{<}$ or e_{\leq} . ◀

6 The NP-hardness of Low

Let Low be the 3-ary relation defined by

$$\text{Low}(x, y, z) := (x < y \wedge z \perp xy) \vee (x < z \wedge y \perp xz).$$

Clearly \perp and $<$ are pp-definable in Low . Note that Low is not preserved by $e_<$ or e_\leq , so $\text{CSP}(P; \text{Low})$ is not covered by the tractability result in Theorem 34. In this section we prove the NP-hardness of $\text{CSP}(P; \text{Low})$.

► **Lemma 43.** *Let us define the relations*

$$\text{Abv}(x, y, z) := (y < x \wedge xy \perp z) \vee (z < x \wedge xz \perp y)$$

$$U(x, y, z) := (y < x \vee z < x) \wedge (y \perp z)$$

Then Abv and U are pp-definable in Low .

Proof. Note that the formula

$$\phi(x, y, z, v) := \exists u \, u \perp v \wedge \text{Low}(u, y, z) \wedge \text{Low}(y, x, v) \wedge \text{Low}(z, x, v)$$

is equivalent to the statement that $v \perp x$ and $y \perp z$ and at least one element of $\{y, z\}$ is smaller than x and at most one element of $\{y, z\}$ is smaller than v

With that in mind one can see that

$$\exists v_1, v_2 \, \phi(x, y, z, v_1) \wedge \phi(v_2, y, z, x)$$

is equivalent to $\text{Abv}(x, y, z)$ and

$$\exists v \, \phi(x, y, z, v)$$

is equivalent to $U(x, y, z)$. ◀

► **Theorem 44.** *Let $a, b \in P$ with $a \perp b$. There is a pp-interpretation of $(\{0, 1\}; \text{1IN3})$ in $(P; \text{Low}, a, b)$. Thus $\text{CSP}(P; \text{Low})$ is NP-hard.*

Proof. Let NAE be the Boolean relation $\{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$. It is easy to see that $\text{Pol}(\{0, 1\}, \text{NAE}, 0, 1)$ is the projection clone **1**. So by Theorem 19 it suffices to show that $(\{0, 1\}; \text{NAE}, 0, 1)$ has a pp-interpretation in $(P; \text{Low}, a, b)$ to prove the Lemma.

Let $D := \{x \in P : \text{Low}(x, a, b)\}$, $D_0 := \{x \in D : x < a\}$, $D_1 := \{x \in D : x < b\}$. Note that $D_0 \perp D_1$. Let $I : D \rightarrow \{0, 1\}$ be given by:

$$I(x) := \begin{cases} 0 & \text{if } x \in D_0 \\ 1 & \text{if } x \in D_1 \end{cases}.$$

Clearly the domain D of I is pp-definable in $(P; \text{Low}, a, b)$. Since the order relation $<$ is pp-definable in Low also the sets D_0 and D_1 are pp-definable. Let $R = \{(x, y, z, t) \in P^4 : (x > y \vee x > z \vee x > t) \wedge \neg(x \leq yzt)\}$. We claim that the relation R is pp-definable in Low . Observe that $(x, y, z, t) \in R$ is equivalent to

$$\exists u, v \, (\text{Abv}(x, u, v) \wedge U(x, y, u) \wedge U(x, z, u) \wedge U(x, t, v))$$

and therefore pp-definable in Low by Lemma 43. By the definition of R we have that $I(c_1, c_2, c_3) \in \text{NAE}$ if and only if $(a, c_1, c_2, c_3) \in R$ and $(b, c_1, c_2, c_3) \in R$. Thus the preimage of NAE is pp-definable in $(P; \text{Low}, a, b)$. ◀

The following lemma gives us an additional characterization of reducts, in which Low is pp-definable.

► **Lemma 45.** *The relation Low is pp-definable in Γ if and only if every binary polymorphism of Γ is dominated.*

Proof. Every dominated function $f : P^2 \rightarrow P$ preserves Low. For the other direction observe that by Lemma 38 we have that f is dominated in the first argument if and only if $f(a_1, b_1) < f(a_2, b_2)$ for all $a_1 < a_2$ and $b_1 \perp b_2$. Note that Lemma 38 also works for non-canonical functions.

So if $f \in \text{Pol}(\Gamma)$ is a binary, not dominated function, there are $a_1 < a_2, b_1 \perp b_2, a'_1 \perp a'_2$ and $b'_1 < b'_2$ such that $f(a_1, b_1) \perp f(a_2, b_2)$ and $f(a'_1, b'_1) \perp f(a'_2, b'_2)$. Hence f violates the relation

$$S(x_1, x_2, y_1, y_2) := (x_1 < x_2 \wedge y_1 \perp y_2) \vee (x_1 \perp x_2 \wedge y_1 < y_2).$$

But the relation S and Low are pp-interdefinable:

$$\begin{aligned} \text{Low}(x, y, z) &\leftrightarrow S(x, y, x, z) \wedge y \perp z \\ S(x_1, x_2, y_1, y_2) &\leftrightarrow \exists u, v, w \ (\text{Low}(x_1, x_2, u) \wedge \text{Abv}(u, x_1, v), \\ &\quad \wedge \text{Low}(u, v, w) \wedge \text{Abv}(w, y_1, v) \wedge \text{Low}(y_1, y_2, w)). \end{aligned}$$

We conclude that f violates Low. ◀

7 Violating the Low relation

We saw in Theorem 34 that $\text{CSP}(\Gamma)$ is tractable if $e_<$ or e_\leq are polymorphisms of Γ . By Theorem 44 we know that $\text{CSP}(\Gamma)$ is NP-complete if Low is pp-definable in Γ . In this section we are going to show that these results already cover all possible reducts where $<$ and \perp are pp-definable.

► **Theorem 46.** *Let Γ be a reduct of \mathbb{P} such that \perp and $<$ are pp-definable in Γ . Then Low is not pp-definable in Γ if and only if $\text{Pol}(\Gamma)$ contains one of the functions $e_<$ or e_\leq .*

Proof. Note that by Theorem 10 Low is not pp-definable in Γ if and only if there is a binary $f \in \text{Pol}(\Gamma)$ violating Low. This means that there are $a, b, c \in P$ such that $a < b \wedge ab \perp c$ and $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$, or $f(a, a) \perp f(b, c)$ and $f(a, a) \perp f(c, b)$.

We have only these two cases since f preserves $\frac{\perp}{<}$ and \perp . We can assume that $a \prec b \prec c$ since otherwise we can find an automorphism $\alpha \in \text{Aut}(\mathbb{P})$ such that $\alpha(a) \prec \alpha(b) \prec \alpha(c)$. Then we consider the map $(x, y) \mapsto f(\alpha^{-1}(x), \alpha^{-1}(y))$ with three elements $\alpha(a)$, $\alpha(b)$ and $\alpha(c)$ instead.

By Theorem 13 we can assume that f is canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. We deal with the two cases in Lemma 48 and Lemma 55 in the following subsections. ◀

► **Notation 47.** *For simplicities sake, a canonical binary function in this section means a function that is canonical as a function from $(P; \leq, \prec)^2 \rightarrow (P; <)$.*

Let $f : P^2 \rightarrow P$ be a function and X, Y, X', Y' be subsets of P such that f is dominated on $X \times Y$ and $X' \times Y'$. We say that f has the same domination on $X \times Y$ and $X' \times Y'$ if f is dominated by the first argument on both $X \times Y$ and $X' \times Y'$, or dominated by the second argument on both $X \times Y$ and $X' \times Y'$. Otherwise, we say that f has the different domination on $X \times Y$ and $X' \times Y'$.

7.1 $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$

The aim of this subsection is to prove the following lemma.

► **Lemma 48.** *Let $f \in \text{Pol}(\Gamma)$ be canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. If $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$ then $\text{Pol}(\Gamma)$ contains $e_<$ or e_\leq .*

We define the following two sets:

- $B_1 := \{x \in P : x > c \wedge x \perp a \wedge x \perp b\},$
- $B_2 := \{x \in P : x > b \wedge x > c\}.$

Let $x, y \in B_1 \cup B_2$. We say that x and y are in the same orbit if $x \in B_i$ and $y \in B_i$ for an $i \in [2]$.

► **Observation 49.** B_1 and B_2 are orbits of $\text{Aut}(P; <, \prec, a, b, c)$. By the homogeneity of $(P; \leq, \prec)$ we can show that $(B_1; \leq, \prec)$, $(B_2; \leq, \prec)$ are isomorphic to $(P; \leq, \prec)$. Further also the union of B_1 and B_2 is an isomorphic copy of $(P; \leq, \prec)$, in which B_1 forms a random filter.

If there is a canonical $g \in \text{Pol}(\Gamma)$ that is not dominated, then Lemma 36 gives us that $e_<$ or e_\leq is in $\text{Pol}(\Gamma)$. So throughout the lemmata and corollaries below in this section, we assume that every binary canonical function in $\text{Pol}(\Gamma)$ is dominated and $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$.

► **Lemma 50.** f is dominated on $B_i \times B_j$ for every $i, j \in [2]$.

Proof. For a contradiction, we assume that f is not dominated on $B_i \times B_j$. Since $(B_i; \leq, \prec)$ and $(B_j; \leq, \prec)$ are isomorphic to $(P; \leq, \prec)$, there are $\alpha : P \rightarrow B_i$ and $\beta : P \rightarrow B_j$ such that α is an isomorphism from $(P; \leq, \prec)$ to $(B_i; \leq, \prec)$ and β is an isomorphism from $(P; \leq, \prec)$ to $(B_j; \leq, \prec)$. Let $g : P^2 \rightarrow P$ be given by $g(x, y) := f(\alpha(x), \beta(y))$. It follows from Observation 49 that g is canonical and is not dominated, a contradiction. ◀

► **Lemma 51.** f has the same domination on all sets $B_i \times B_j$, $i, j \in [2]$.

Proof. We claim that f has the same domination on $B_1 \times B_k$ and $B_2 \times B_k$ for any $k \in [2]$. For a contradiction, we assume that f does not have the same domination $B_1 \times B_k$ and $B_2 \times B_k$. Without loss of generality we can assume that f is dominated by the first argument on $B_1 \times B_k$ and dominated by the second argument on $B_2 \times B_k$. Let $x, y \in B_1, z, t \in B_2$ be such that $x < y \wedge y < z \wedge x \perp t$. Let $x', y', z', t' \in B_k$ be such that $x' \perp t' \wedge y' < z' \wedge z' < t'$. Since f is dominated by the first argument on $B_1 \times B_k$, we have $f(x, x') < f(y, y')$. Since f is dominated by the second argument on $B_2 \times B_k$, we have $f(z, z') < f(t, t')$. Since f preserves $<$, we have $f(y, y') < f(z, z')$. Thus $f(x, x') < f(t, t')$, a contradiction to the fact that f preserves \perp .

By considering the map $(x, y) \mapsto f(y, x)$ we have that f has the same domination on $B_k \times B_1$ and $B_k \times B_2$ for every $k \in [2]$. This implies that f has the same dominations on all products $B_i \times B_j$, $i, j \in [2]$. ◀

In the rest of this section, we assume that f is dominated by the first argument on $B_i \times B_j$ for every $i, j \in [2]$. The other case can be reduced to this case by considering the map $(x, y) \mapsto f(y, x)$.

► **Lemma 52.** Let $u, v \in B_1$ and $u' \in B_2, v' \in B_1$ be such that $u < v \vee u \perp v$. Then $f(u, u') \perp f(v, v')$.

Proof. First, we claim that $f(u, u') > f(v, v') \vee f(u, u') \perp f(v, v')$. For a contradiction, we assume that $f(u, u') \leq f(v, v')$. Since f preserves $<$, we have $f(c, b) < f(u, u')$. Therefore $f(a, a) < f(c, b) < f(u, u') < f(v, v')$, a contradiction to the \perp -preservation of f . Thus the claim follows.

The proof is completed by showing that $f(u, u') > f(v, v')$ is impossible. For a contradiction, we assume that $f(u, u') > f(v, v')$. Let $s, t \in B_1$ be such that $s \perp t \wedge s < v \wedge u < t$. Let $s' \in B_1, t' \in B_2$ be such that $s' \perp t'$. By the domination of f , we have $f(s, s') < f(v, v') \wedge f(u, u') < f(t, t')$. It follows from $f(u, u') > f(v, v')$, we have $f(s, s') < f(t, t')$, a contradiction to \perp -preservation of f . \blacktriangleleft

► **Lemma 53.** *Let $u, v \in B_1$ be such that $u \perp v$. Then for every $u', v' \in B_1 \cup B_2$, we have $f(u, u') \perp f(v, v')$.*

Proof. For a contradiction, we assume that $\neg(f(u, u') \perp f(v, v'))$. Without loss of generality, we assume that $f(u, u') \leq f(v, v')$. Let $s, t \in B_1$ be such that $s < u \wedge v < t \wedge s \perp t$. Let $s', t' \in B_1 \cup B_2$ be such that $s' \perp t'$, s', u' are in the same orbit and t', v' are in the same orbit. By the domination of f , we have $f(s, s') < f(u, u') \wedge f(v, v') < f(t, t')$. Since $f(u, u') < f(v, v')$, we have $f(s, s') < f(t, t')$, a contradiction to the \perp -preservation of f . \blacktriangleleft

► **Lemma 54.** *Let $u, v \in B_1$ and $u', v' \in B_1 \cup B_2$ be such that $u < v$. Then $f(u, u') < f(v, v') \vee f(u, u') \perp f(v, v')$.*

Proof. For a contradiction, we assume that $f(v, v') \leq f(u, u')$. Let $s, t \in B_1$ be such that $t < v \wedge u < s \wedge s \perp t$. Let $s', t' \in B_1 \cup B_2$ be such that $s' \perp t'$, s', u' are in the same orbit, and t', v' are in the same orbit. By the domination of f , we have $f(t, t') < f(v, v') \wedge f(u, u') < f(s, s')$. Since $f(v, v') < f(u, u')$, we have $f(t, t') < f(s, s')$, a contradiction to the \perp -preservation of f . \blacktriangleleft

Proof of Lemma 48. We are going to show that $\text{Pol}(\Gamma)$ contains a function that behaves like $e_<$ or like e_\leq by checking the conditions of Lemma 42.

So let $\bar{a}, \bar{b} \in P^k$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$. We set $Y := \{b_i : b_i \geq b_p\}, Z := \{b_i : \neg(b_i \geq b_p)\}$. By definition we have $b_q \in Z$. By the homogeneity of \mathbb{P} , there is $\alpha \in \text{Aut}(\mathbb{P})$ such that $\alpha(Y) \subseteq B_2$ and $\alpha(Z) \subseteq B_1$. Let $\beta \in \text{Aut}(\mathbb{P})$ such that $\beta(\{a_i : i \in [k]\}) \subseteq B_1$. Let $g(x, y) := f(\beta(x), \alpha(y))$. Clearly, $g \in \text{Pol}(\Gamma)$.

By Lemma 52 we have that $g(a_p, b_p) \perp g(a_q, b_q)$. Further we know by Lemma 54 that $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i < a_j$. By Lemma 53 we know that $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i \perp a_j$. So the conditions of Lemma 42 are satisfied. Hence $e_<$ or e_\leq is a polymorphism of Γ . \blacktriangleleft

7.2 $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$

The aim of this section is to prove the following.

► **Lemma 55.** *Let $f \in \text{Pol}(\Gamma)$ be canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. If $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$, then $\text{Pol}^{(2)}(\Gamma)$ contains $e_<$ or e_\leq .*

We define the following sets.

$$\begin{aligned} B_1 &:= \{x \in P : a < x < b \wedge x \perp c\} \\ B_2 &:= \{x \in P : x < b \wedge x < c \wedge x \perp a \wedge x \prec a\}. \end{aligned}$$

Throughout the lemmata and corollaries below in this section, we assume that every binary canonical function in Γ is dominated and $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$.

Observe that by the homogeneity of $(P; \leq; \prec)$ and the back-and-forth argument, we can show that $(B_1 \cup B_2; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$, with B_2 being a random filter. For every two k -tuples \bar{x} and \bar{y} in B_i^k , \bar{x} and \bar{y} are in the same orbit of $\text{Aut}(\mathbb{P})$ if and only if \bar{x} and \bar{y} are in the same orbit of $\text{Aut}(P; a, b, c)$.

► **Lemma 56.** *f has the same domination on sets $B_i \times B_j, i, j \in [2]$.*

Proof. This lemma can be shown as in Lemma 50 and Lemma 51. ◀

In the rest of this section we assume that f is dominated by the first argument on $B_i \times B_j$ for every $i, j \in \{1, 2\}$. Similarly, to Lemma 52, we have the following.

► **Lemma 57.** *Let $u, v \in B_1$ and $u' \in B_1, v' \in B_2$ be such that $u < v \vee u \perp v$. Then $f(u, u') \perp f(v, v')$.*

Proof. First we prove that $f(v, v') < f(u, u') \vee f(v, v') \perp f(u, u')$. For a contradiction we assume that $f(u, u') \leq f(v, v')$. Since $a < u \wedge a < u'$, we have $f(a, a) < f(u, u')$. Since $v < b \wedge v' < c$, we have $f(v, v') < f(b, c)$. Thus $f(a, a) < f(b, c)$, a contradiction to the fact that $f(a, a) \perp f(b, c)$. Thus $f(v, v') < f(u, u') \vee f(v, v') \perp f(u, u')$.

The proof is completed by showing that $f(u, u') > f(v, v')$ is impossible. For a contradiction, we assume that $f(u, u') > f(v, v')$. Let $s, t \in B_1$ be such that $s \perp t \wedge s < v \wedge u < t$. Let $s' \in B_2, t' \in B_1$ be such that $s' \perp t'$. By the domination of f , we have $f(s, s') < f(v, v') \wedge f(u, u') < f(t, t')$. It follows from $f(u, u') > f(v, v')$, we have $f(s, s') < f(t, t')$, a contradiction to \perp -preservation of f . ◀

► **Lemma 58.** *Let $u, v \in B_1$ be such that $u \perp v$. Then for every $u', v' \in B_1 \cup B_2$, we have $f(u, u') \perp f(v, v')$.*

Proof. analogous to Lemma 53. ◀

► **Lemma 59.** *Let $u, v \in B_1$ and $u', v' \in B_1 \cup B_2$ be such that $u < v$. Then $f(u, u') < f(v, v') \vee f(u, u') \perp f(v, v')$.*

Proof. analogous to Lemma 54. ◀

Proof of Lemma 55. We are again going to show that $\text{Pol}(\Gamma)$ contains a function that behaves like $e_{<}$ or like e_{\leq} by checking the conditions of Lemma 42.

So let $\bar{a}, \bar{b} \in P^k$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$. We set $Y := \{b_i : b_i \geq b_p\}, Z := \{b_i : \neg(b_i \geq b_p)\}$. By definition we have $b_q \in Z$. By the homogeneity of \mathbb{P} , there is $\alpha \in \text{Aut}(\mathbb{P})$ such that $\alpha(Y) \subseteq B_1$ and $\alpha(Z) \subseteq B_2$. Let $\beta \in \text{Aut}(\mathbb{P})$ such that $\beta(\{a_i : i \in [k]\}) \subseteq B_1$. Let $g(x, y) := f(\beta(x), \alpha(y))$. Clearly, $g \in \text{Pol}(\Gamma)$.

By Lemma 52 we have that $g(a_p, b_p) \perp g(a_q, b_q)$. Further we know by Lemma 54 that $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i < a_j$. By Lemma 53 we know that $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i \perp a_j$. So the conditions of Lemma 42 are satisfied. Hence $e_{<}$ or e_{\leq} is a polymorphism of Γ . ◀

8 The NP-hardness of Betw, Sep and Cycl

By Corollary 22 we are now left with the cases where $\text{End}(\Gamma)$ is equal to one of the monoids $\langle \uparrow \rangle$, $\langle \circ \rangle$ or $\langle \uparrow, \circ \rangle$.

We are going to deal with all these remaining cases in this section. Interestingly, we can treat them all similarly: By fixing finitely many constants c_1, \dots, c_n on Γ we obtain definable subsets of $(\Gamma, c_1, \dots, c_n)$ on which $<$ and Low are pp-definable. This enables us to reduce every every such case to the NP-completeness of Low.

► **Lemma 60.** *Let $u, v \in P$ with $u < v$. Then the relations $<$ and Low are pp-definable in $(P, \text{Betw}, \perp, u, v)$.*

Proof. It is easy to verify that there is a pp-definition of the order relation by the following equivalence:

$$x < y \leftrightarrow \exists a, b (\text{Betw}(x, y, a) \wedge \text{Betw}(y, a, b) \wedge \text{Betw}(u, v, a) \wedge \text{Betw}(v, a, b)).$$

The two maps $e_{<} : P^2 \rightarrow P$ and $e_{\leq} : P^2 \rightarrow P$ do not preserve Betw, since for every triple $\bar{a} = (a_1, a_2, a_3)$ with $a_1 < a_2 < a_3$ and $\bar{b} = (b_1, b_2, b_3)$ with $b_1 > b_2 > b_3$, the image of (\bar{a}, \bar{b}) forms an antichain.

By Theorem 46 we have that Low is pp-definable in $(P, \text{Betw}, \perp, u, v)$. ◀

► **Theorem 61.** *Let Γ be a reduct of \mathbb{P} such that $\text{End}(\Gamma) = \langle \uparrow \rangle$. Then there are constants $u, v, w, t \in P$ such that $(\{0, 1\}, 1\text{IN}3)$ is pp-interpretable in (Γ, u, v, w, t) . Hence $\text{CSP}(\Gamma)$ is NP-complete.*

Proof. Note that the betweenness relation Betw is an orbit of $\text{End}(\Gamma) = \langle \uparrow \rangle$ on P^3 . Now Theorem 10 implies that Betw is primitively positive definable in Γ . For the same reason \perp is pp-definable in Γ . By Lemma 60 there is pp-definition of Low in (Γ, u, v) . By Theorem 44 we can find a pp-interpretation of $(\{0, 1\}, 1\text{IN}3)$ in (Γ, u, v, w, t) , where w, t are two additional constants. Hence $\text{CSP}(\Gamma)$ is NP-complete. ◀

For the case where $\text{End}(\Gamma) = \langle \circ \rangle$, we first need the following lemma:

► **Lemma 62.** *Let c, d be two constants in P such that $c < d$. Then there is a pp-interpretation of $(P; \text{Low})$ in $(P; \text{Cycl}, c, d)$.*

Proof. Let $X := \{x \in P : c < x < d\}$. By using back-and-forth argument one can show easily that $(P; <)$ and $(X; <|_X)$ are isomorphic. We first show that X (as a unary predicate) and $<|_X$ are pp-definable in $(P; \text{Cycl}, c, d)$. It is easy to verify that the set X can be defined in $(P; \text{Cycl}, c, d)$ by $\phi(x) := \text{Cycl}(c, x, d)$ and that $x <|_X y \leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Cycl}(c, x, y)$. Now a pp-interpretation of $(P; <, \text{Cycl})$ in $(P; \text{Cycl}, c, d)$ is simply given by the identity on X .

By Lemma 23 we have that \perp is pp-definable in $(P; <, \text{Cycl})$. It is easy to verify that $e_{<}$ and e_{\leq} do not preserve Cycl. Therefore, by Theorem 46, Low is pp-definable in $(P; <, \text{Cycl})$, which concludes the proof of the Lemma. ◀

► **Theorem 63.** *Let Γ be a reduct of \mathbb{P} such that $\text{End}(\Gamma) = \langle \circ \rangle$. Then there are constants $a, b, c, d \in P$ such that $(\{0, 1\}, 1\text{IN}3)$ is pp-interpretable in (Γ, a, b, c, d) . Hence $\text{CSP}(\Gamma)$ is NP-complete.*

Proof. The cyclic order relation Cycl is an orbit of $\text{End}(\Gamma) = \langle \circ \rangle$ on P^3 . So Theorem 10 implies that Cycl is primitively positive definable in Γ . By Lemma 62 there is pp-definition of Low in (Γ, c, d) with $c < d$. By Theorem 44 we can find a pp-interpretation of $(\{0, 1\}, 1\text{IN}3)$ in (Γ, a, b, c, d) , where a, b are two additional constants. Hence $\text{CSP}(\Gamma)$ is NP-complete. ◀

In the following, we prove the NP-hardness of $\text{CSP}(P; \text{Sep})$ by using the same proof idea as the proof of NP-hardness of $\text{CSP}(P; \text{Cycl})$ in Section 8.

► **Lemma 64.** *Let c, d, u be constants in P such that $c < d < u$. Then $(P; \text{Low})$ has a pp-interpretation in $(P; \text{Sep}, c, d, u)$.*

Proof. Let $X := \{x \in P : d < x < u\}$. By using a back-and-forth argument, one can show easily that $(X; \leq)$ and \mathbb{P} are isomorphic. Similarly as in the proof of Theorem 63, X and $<|_X$ are pp definable in $(P; \text{Sep}, c, d, u)$ as follows.

The set X can be defined by the formula $\phi(x) := \text{Sep}(c, d, x, u)$, and $<|_X$ can be defined by $x <|_X y \Leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Sep}(c, d, x, y)$. Also $\text{Cycl}(x, y, z)|_X$ can be defined by the primitive positive formula $\phi(x) \wedge \phi(y) \wedge \phi(z) \wedge \text{Sep}(c, x, y, z)$

So a pp-interpretation of $(P; <, \text{Cycl})$ in $(P; \text{Sep}, c, d, u)$ is simply given by the identity, restricted to X . By Lemma 62, Low is pp-definable in $(P; <, \text{Cycl})$, which concludes the proof of the Lemma. ◀

► **Theorem 65.** *Let Γ be a reduct of \mathbb{P} such that $\text{End}(\Gamma) = \overline{\langle \uparrow, \circ \rangle}$. Then there are constants $a, b, c, d, u \in P$ such that $(\{0, 1\}, 1\text{IN}3)$ is pp-interpretable in (Γ, a, b, c, d, u) . Hence $\text{CSP}(\Gamma)$ is NP-complete.*

Proof. The relation Sep is an orbit of $\text{End}(\Gamma) = \overline{\langle \uparrow, \circ \rangle}$ on P^3 . So Theorem 10 implies that Sep is primitively positive definable in Γ . By Lemma 64 there is pp-definition of Low in (Γ, c, d, u) with $c < d < u$. By Theorem 44 we can find a pp-interpretation of $(\{0, 1\}, 1\text{IN}3)$ in (Γ, a, b, c, d, u) , where a, b are two additional constants. Hence $\text{CSP}(\Gamma)$ is NP-complete. ◀

9 Main Results

In this section we complete the proof of the complexity dichotomy for the Poset-SAT(Φ) problems that we announced in Theorem 5 and that we reformulated as CSPs on the reducts of the random poset \mathbb{P} . We have proven an even stronger dichotomy that remains interesting even if $P=NP$. This dichotomy regards model-theoretic properties of the reducts of \mathbb{P} and can be also stated in terms of universal-algebra by what we saw in Section 3.5. We will phrase it in Theorem 67.

9.1 An algebraic dichotomy

Let Γ be a reduct of \mathbb{P} and Δ be its model-complete core. Throughout this paper we have studied the question whether there is a pp-interpretation of the structure $(\{0, 1\}; 1\text{IN}3)$ in Δ , extended by finitely many constants or not.

By Theorem 18 and Theorem 19 we know that this fact can be elegantly described with the help of topological clones. We sum up our results and show that - for reducts of the random poset - we can also give an additional characterization by weak near unanimity polymorphisms (modulo endomorphism). First we are going to look in detail at the case, where $<$ and \perp are pp-definable.

► **Lemma 66.** *Let Γ be a reduct of \mathbb{P} in which $<$ and \perp are pp-definable. Then the following are equivalent:*

1. *There is a binary $f \in \text{Pol}(\Gamma)$ which is not dominated.*
2. *The relation Low is not pp-definable in Γ .*
3. *$e_<$ or e_\leq is a polymorphism of Γ .*

4. There is a binary $f \in \text{Pol}(\Gamma)$ and endomorphisms $e_1, e_2 \in \text{End}(\Gamma)$ such that

$$e_1(f(x, y)) = e_2(f(y, x))$$

5. For all $c_1, \dots, c_n \in \Gamma$ there is no clone homomorphism from $\text{Pol}(\Gamma, c_1, \dots, c_n)$ onto **1**.
 6. For all $c_1, \dots, c_n \in \Gamma$ there is no continuous clone homomorphism from $\text{Pol}(\Gamma, c_1, \dots, c_n)$ onto **1**.
 7. There is no pp-interpretation of $(\{0, 1\}; 1\text{IN}3)$ in any expansion of Γ by finitely many constants.

Proof.

The equivalences of the points (5)-(7) hold for all ω -categorical structures Γ and were discussed in Theorem 19.

(1) \leftrightarrow (2) This is the statement of Lemma 45.

(2) \rightarrow (3): This is the statement of Theorem 46.

(3) \rightarrow (4): Set $f = e_{<}$ respectively $f = e_{\leq}$.

(4) \rightarrow (5): If there are $e_1, e_2, f \in \text{Pol}(\Gamma)$ satisfying the equation $e_1(f(x, y)) = e_2(f(y, x))$ then there are also such polymorphisms fixing finitely many elements c_1, \dots, c_n . This is true for all ω -categorical cores, see Lemma 82 of [17]. It follows that there is no clone homomorphism from $\text{Pol}(\Gamma, c_1, \dots, c_n)$ onto **1**.

(7) \rightarrow (2): This follows from the contraposition of Theorem 44. \blacktriangleleft

With Lemma 66 we are now able to show the following Theorem.

► **Theorem 67.** *Let Γ be a reduct of \mathbb{P} and let Δ be the model-complete core of Γ . Then the following are equivalent:*

1. There is a binary $f \in \text{Pol}(\Delta)$ and endomorphisms $e_1, e_2 \in \text{End}(\Delta)$ such that

$$e_1(f(x, y)) = e_2(f(y, x))$$

or there is a ternary $f \in \text{Pol}(\Delta)$ and endomorphisms $e_1, e_2, e_3 \in \text{End}(\Delta)$ such that

$$e_1(f(x, x, y)) = e_2(f(x, y, x)) = e_3(f(y, x, x)).$$

2. There is a pseudo Siggers polymorphism, i.e. a function $f \in \text{Pol}(\Delta)^{(6)}$ and endomorphism $e_1, e_2 \in \text{End}(\Delta)$ such that

$$e_1(f(x, y, x, z, y, z)) = e_2(f(y, x, z, x, z, y)).$$

3. For all $c_1, \dots, c_n \in \Delta$ there is no clone homomorphism from $\text{Pol}(\Delta)$ onto **1**.
 4. For all $c_1, \dots, c_n \in \Delta$ there is no continuous clone homomorphism from $\text{Pol}(\Delta)$ onto **1**.
 5. There is no pp-interpretation of $(\{0, 1\}; 1\text{IN}3)$ in any expansion of Δ by finitely many constants.

Proof. First of all we remark that the equivalence of the points (2)-(5) holds for all ω -categorical core structures and was discussed in Theorem 19.

In Theorem 21 we saw that the model-complete core Δ is either equal to Γ or a reduct of $(\mathbb{Q}, <)$ or $(\omega, =)$.

Suppose the core Δ is a reduct of $(\mathbb{Q}, <)$ or $(\omega, =)$. We know from the analysis of temporal constraint satisfaction problems that then the statement is true: By Theorem 10.1.1. in [3] there is no pp-interpretation of $(\{0, 1\}; 1\text{IN}3)$ in Δ , if and only if an equation $e_1(f(x, x, y)) = e_2(f(x, y, x)) = e_3(f(y, x, x))$ holds in $\text{Pol}(\Delta)$.

So let $\Gamma = \Delta$. By Lemma 66 the equivalence (1) \leftrightarrow (4) holds when $<$ and \perp are pp-definable in Γ . In the remaining cases $\text{End}(\Gamma)$ is equal to $\langle \uparrow \rangle$, $\langle \circ \rangle$ or $\langle \uparrow, \circ \rangle$ and we have a pp-interpretation of $(\{0, 1\}; 1\text{IN}3)$ in an extension of Γ with finitely many constants Theorems 44, 61, 63 and 65. \blacktriangleleft

9.2 A complexity dichotomy

For the complexity of the CSPs of reducts of \mathbb{P} that are model-complete we have proven the following dichotomy:

- **Theorem 68.** *Let Γ be a reduct of \mathbb{P} in a finite relational language and a model-complete core. Under the assumption $P \neq NP$ either*
- *one of the relations Low, Betw, Cycl, Sep is pp-definable in Γ and $\text{CSP}(\Gamma)$ is NP-complete or*
 - *$\text{CSP}(\Gamma)$ is tractable.*

Proof. If Low, Betw, Cycl or Sep is pp-definable in Γ , the $\text{CSP}(\Gamma)$ is NP-complete by the Theorems 61, 65, 63 and 44.

By Theorem 21 the only remaining case is the one, where $<$ and \perp are pp-definable, but Low is not. In this case $e_<$ or e_\perp is a polymorphism of Γ by Theorem 46. Theorem 34 then implies that the problem is tractable. ◀

- **Corollary 69.** *Let Γ be a reduct of \mathbb{P} in a finite relational language. Under the assumption $P \neq NP$ the problem $\text{CSP}(\Gamma)$ is either NP-complete or solvable in polynomial time. Further the “meta-problem” of deciding whether a given problem $\text{CSP}(\Gamma)$ is tractable or NP-complete, is decidable.*

Proof. By Theorem 21 we know that either Γ is a model-complete core or $g_<$ or g_\perp are endomorphisms of Γ . In the first case the dichotomy holds by Theorem 68, in the second case Γ is homomorphically equivalent to a reduct of $(\mathbb{Q}, <)$ and the dichotomy holds by the result in [6] respectively [5].

The main result in [12] imply that it is decidable if the relations $<$, \perp , Low, Betw, Cycl or Sep are pp-definable in Γ . By Lemma 24 the question whether Γ is model-complete core or not is then also decidable. By Theorem 68 and Corollary 52 of [6] we have that the meta-problem is decidable. ◀

We finish with an algebraic version of our dichotomy that is a direct implication of Theorem 67:

- **Corollary 70.** *Let Γ be a reduct of \mathbb{P} in a finite relational language and let Δ be its model complete core. Under the assumption $P \neq NP$ either*
- *$\text{CSP}(\Gamma)$ is NP-complete and all finite structures are pp-interpretable in Δ , extended by finitely many constant, or*
 - *$\text{CSP}(\Gamma)$ is tractable and the conditions (1)-(6) in Theorem 67 hold.* ◻

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